Quantification of Model Risk in Quadratic Hedging in Finance

Catherine Daveloose, Asma Khedher and Michèle Vanmaele

Abstract In this paper the effect of the choice of the model on partial hedging in incomplete markets in finance is estimated. In fact we compare the quadratic hedging strategies in a martingale setting for a claim when two models for the underlying stock price are considered. The first model is a geometric Lévy process in which the small jumps might have infinite activity. The second model is a geometric Lévy process where the small jumps are replaced by a Brownian motion which is appropriately scaled. The hedging strategies are related to solutions of backward stochastic differential equations with jumps which are driven by a Brownian motion and a Poisson random measure. We use this relation to prove that the strategies are robust towards the choice of the model for the market prices and to estimate the model risk.

Keywords Lévy models · Quadratic hedging · Model risk · Robustness · BSDEJs

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1 Introduction

When jumps are present in the stock price model, the market is in general incomplete and there is no self-financing hedging strategy which allows to attain the contingent claim at maturity. In other words, one cannot eliminate the risk completely. However

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it is possible to find 'partial' hedging strategies which minimise some risk. One way to determine these 'partial' hedging strategies is to introduce a subjective criterion according to which strategies are optimised.

In the present paper, we consider two types of quadratic hedging strategies. The first, called *risk-minimising* (RM) strategy, is replicating the option's payoff, but it is not self-financing (see, e.g., [19]). In such strategies, the hedging is considered under a *risk-neutral measure* or *equivalent martingale measure*. The aim is to minimise the risk process, which is induced by the fact that the strategy is not self-financing, under this measure. In the second approach, called *mean-variance hedging* (MVH), the strategy is self-financing and the quadratic hedging error at maturity is minimised in mean square sense (see, e.g., [19]). Again a risk-neutral setting is assumed.

The aim in this paper is to investigate whether these quadratic hedging strategies (RM and MVH) in incomplete markets are robust to the variation of the model. Thus we consider two geometric Lévy processes to model the asset price dynamics. The first model $(S_t)_{t\in[0,T]}$ is driven by a Lévy process in which the small jumps might have infinite activity. The second model $(S_t^\varepsilon)_{t\in[0,T]}$ is driven by a Lévy process in which we replace the jumps with absolute size smaller than $\varepsilon > 0$ by an appropriately scaled Brownian motion. The latter model $(S_t^\varepsilon)_{t\in[0,T]}$ converges to the first one in an L^2-sense when ε goes to 0. The aim is to study whether similar convergence properties hold for the corresponding quadratic hedging strategies.

Geometric Lévy processes describe well realistic asset price dynamics and are well established in the literature (see e.g., [5]). Moreover, the idea of shifting from a model with small jumps to another where these variations are represented by some appropriately scaled continuous component goes back to [2]. This idea is interesting from a simulation point of view. Indeed, the process $(S_t^\varepsilon)_{t\in[0,T]}$ contains a compound Poisson process and a scaled Brownian motion which are both easy to simulate. Whereas it is not easy to simulate the infinite activity of the small jumps in the process $(S_t)_{t\in[0,T]}$ (see [5] for more about simulation of Lévy processes).

The interest of this paper is the *model risk*. In other words, from a modelling point of view, we may think of two financial agents who want to price and hedge an option. One is considering $(S_t)_{t\in[0,T]}$ as a model for the price process and the other is considering $(S_t^\varepsilon)_{t\in[0,T]}$. Thus the first agent chooses to consider infinitely small variations in a discontinuous way, i.e. in the form of infinitely small jumps of an infinite activity Lévy process. The second agent observes the small variations in a continuous way, i.e. coming from a Brownian motion. Hence the difference between both market models determines a type of model risk and the question is whether the pricing and hedging formulas corresponding to $(S_t^\varepsilon)_{t\in[0,T]}$ converge to the pricing and hedging formulas corresponding to $(S_t)_{t\in[0,T]}$ when ε goes to zero. This is what we intend in the sequel by *robustness* or *stability* study of the model.

In this paper we focus mainly on the RM strategies. These strategies are considered under a martingale measure which is equivalent to the historical measure. Equivalent martingale measures are characterised by the fact that the discounted asset price processes are martingales under these measures. The problem we are facing is that the martingale measure is dependent on the choice of the model. Therefore it is clear that, in this paper, there will be different equivalent martingale measures for the two considered price models. Here we emphasise that for the robustness study, we come back to the common underlying physical measure.

Besides, since the market is incomplete, we will also have to identify which equivalent martingale measure, or measure change, to apply. In particular, we discuss some specific martingale measures which are commonly used in finance and in electricity markets: the Esscher transform, the minimal entropy martingale measure, and the minimal martingale measure. We prove some common properties for the mentioned martingale measures in the exponential Lévy setting in addition to those shown in [4, 6].

To perform the described stability study, we follow the approach in [8] and we relate the RM hedging strategies to backward stochastic differential equations with jumps (BSDEJs). See e.g. [7, 9] for an overview about BSDEs and their applications in hedging and in nonlinear pricing theory for incomplete markets.

Under some conditions on the parameters of the stock price process and of the martingale measure, we investigate the robustness to the choice of the model of the value of the portfolio, the amount of wealth, the cost and gain process in a RM strategy. The amount of wealth and the gain process in a MVH strategy coincide with those in the RM strategy and hence the convergence results will immediately follow. When we assume a fixed initial portfolio value to set up a MVH strategy we derive a convergence rate for the loss at maturity.

The BSDEJ approach does not provide a robustness result for the *optimal number* of risky assets in a RM strategy as well as in a MVH strategy. In [6] convergence rates for those optimal numbers and other quantities, such as the delta and the amount of wealth, are computed using Fourier transform techniques.

The paper is organised as follows: in Sect. 2 we introduce the notations, define the two martingale models for the stock price, and derive the corresponding BSDEJs for the value of the discounted RM hedging portfolio. In Sect. 3 we study the stability of the quadratic hedging strategies towards the choice of the model and obtain convergence rates. In Sect. 4 we conclude.

2 Quadratic Hedging Strategies in a Martingale Setting for Two Geometric Lévy Stock Price Models

Assume a finite time horizon $T > 0$. The first considered stock price process is determined by the process $L = (L_t)_{t\in[0,T]}$ which denotes a Lévy process in the filtered complete probability space $(\Omega, \mathscr{F}, \mathbb{F}, \mathbb{P})$ satisfying the usual hypotheses as defined in [18]. We work with the càdlàg version of the given Lévy process. The characteristic triplet of the Lévy process L is denoted by (a, b^2, ℓ). We consider a stock price modelled by a geometric Lévy process, i.e. the stock price is given by $S_t = S_0 e^{L_t}$, $\forall t \in [0, T]$, where $S_0 > 0$. Let $r > 0$ be the risk-free instantaneous interest rate. The value of the corresponding riskless asset equals e^{rt} for any time

$t \in [0, T]$. We denote the discounted stock price process by $\hat{S}$. Hence at any time $t \in [0, T]$ it equals

$$\hat{S}_t = \mathrm{e}^{-rt} S_t = S_0 \mathrm{e}^{-rt} \mathrm{e}^{L_t}.$$

It holds that

$$\mathrm{d}\hat{S}_t = \hat{S}_t \hat{a} \mathrm{d}t + \hat{S}_t b \mathrm{d}W_t + \hat{S}_t \int_{\mathbb{R}_0} (\mathrm{e}^z - 1) \widetilde{N}(\mathrm{d}t, \mathrm{d}z), \tag{1}$$

where W is a standard Brownian motion independent of the compensated jump measure $\widetilde{N}$ and

$$\hat{a} = a - r + \frac{1}{2} b^2 + \int_{\mathbb{R}_0} \left(\mathrm{e}^z - 1 - z 1_{\{|z|<1\}} \right) \ell(\mathrm{d}z).$$

It is assumed that $\hat{S}$ is not deterministic and arbitrage opportunities are excluded (cfr. [21]). The aim of this paper is to study the stability of quadratic hedging strategies in a martingale setting towards the choice of the model. Since the equivalent martingale measure is determined by the market model, we also have to take into account the robustness of the risk-neutral measures. Therefore we consider the case where $\mathbb{P}$ is not a risk-neutral measure, or in other words $\hat{a} \neq 0$ so that $\hat{S}$ is not a $\mathbb{P}$-martingale. Then, a change of measure, specifically determined by the market model (1), will have to be performed to obtain a martingale setting. Let us denote a martingale measure which is equivalent to the historical measure $\mathbb{P}$ by $\widetilde{\mathbb{P}}$. We consider martingale measures that belong to the class of structure preserving martingale measures, see [14]. In this case, the Lévy triplet of the driving process L under $\widetilde{\mathbb{P}}$ is denoted by $(\tilde{a}, b^2, \tilde{\ell})$. Theorem III.3.24 in [14] states conditions which are equivalent to the existence of a parameter $\Theta \in \mathbb{R}$ and a function $\rho(z; \Theta)$, $z \in \mathbb{R}$, such that

$$\int_{\{|z|<1\}} |z\,(\rho(z; \Theta) - 1)|\, \ell(\mathrm{d}z) < \infty, \tag{2}$$

and such that

$$\tilde{a} = a + b^2 \Theta + \int_{\{|z|<1\}} z\,(\rho(z; \Theta) - 1)\, \ell(\mathrm{d}z) \quad \text{and} \quad \tilde{\ell}(\mathrm{d}z) = \rho(z; \Theta) \ell(\mathrm{d}z). \tag{3}$$

For $\hat{S}$ to be a martingale under $\widetilde{\mathbb{P}}$, the parameter Θ should guarantee the following equation

$$\hat{a}_0 = \tilde{a} - r + \frac{1}{2} b^2 + \int_{\mathbb{R}_0} \left(\mathrm{e}^z - 1 - z 1_{\{|z|<1\}} \right) \tilde{\ell}(\mathrm{d}z) = 0. \tag{4}$$

From now on we denote the solution of Eq. (4)—when it exists—by Θ_0 and the equivalent martingale measure by $\widetilde{\mathbb{P}}_{\Theta_0}$. Notice that we obtain different martingale measures $\widetilde{\mathbb{P}}_{\Theta_0}$ for different choices of the function $\rho(.\,; \Theta_0)$. In the next section we present some known martingale measures for specific functions $\rho(.\,;\Theta_0)$ and specific parameters Θ_0 which solve (4).

The relation between the original measure $\mathbb{P}$ and the martingale measure $\widetilde{\mathbb{P}}_{\Theta_0}$ is given by

$$\frac{\mathrm{d}\widetilde{\mathbb{P}}_{\Theta_0}}{\mathrm{d}\mathbb{P}}\Big|_{\mathscr{F}_t} = \exp\Bigg(b\Theta_0 W_t - \frac{1}{2}b^2{\Theta_0}^2 t + \int_0^t \int_{\mathbb{R}_0} \log\left(\rho(z;\Theta_0)\right)\widetilde{N}(\mathrm{d}s,\mathrm{d}z) \\ + t\int_{\mathbb{R}_0}\left(\log\left(\rho(z;\Theta_0)\right) + 1 - \rho(z;\Theta_0)\right)\ell(\mathrm{d}z)\Bigg).$$

From the Girsanov theorem (see e.g. Theorem 1.33 in [17]) we know that the processes W^{Θ_0} and $\widetilde{N}^{\Theta_0}$ defined by

$$\begin{aligned} \mathrm{d}W_t^{\Theta_0} &= \mathrm{d}W_t - b\Theta_0\mathrm{d}t, \\ \widetilde{N}^{\Theta_0}(\mathrm{d}t,\mathrm{d}z) &= N(\mathrm{d}t,\mathrm{d}z) - \rho(z;\Theta_0)\ell(\mathrm{d}z)\mathrm{d}t = \widetilde{N}(\mathrm{d}t,\mathrm{d}z) + \left(1-\rho(z;\Theta_0)\right)\ell(\mathrm{d}z)\mathrm{d}t, \end{aligned} \tag{5}$$

for all $t \in [0,T]$ and $z \in \mathbb{R}_0$, are a standard Brownian motion and a compensated jump measure under $\widetilde{\mathbb{P}}_{\Theta_0}$. Moreover we can rewrite (1) as

$$\mathrm{d}\hat{S}_t = \hat{S}_t b\mathrm{d}W_t^{\Theta_0} + \hat{S}_t\int_{\mathbb{R}_0}(\mathrm{e}^z - 1)\widetilde{N}^{\Theta_0}(\mathrm{d}t,\mathrm{d}z). \tag{6}$$

We consider an $\mathscr{F}_T$-measurable and square integrable random variable H_T which denotes the payoff of a contract. The discounted payoff equals $\hat{H}_T = \mathrm{e}^{-rT}H_T$. In case the discounted stock price process is a martingale, both, the mean-variance hedging (MVH) and the risk-minimising strategy (RM) are related to the Galtchouk-Kunita-Watanabe (GKW) decomposition, see [11]. In the following we recall the GKW-decomposition of the $\mathscr{F}_T$-measurable and square integrable random variable $\hat{H}_T$ under the martingale measure $\widetilde{\mathbb{P}}_{\Theta_0}$

$$\hat{H}_T = \widetilde{\mathbb{E}}^{\Theta_0}[\hat{H}_T] + \int_0^T \xi_s^{\Theta_0}\mathrm{d}\hat{S}_s + \mathscr{L}_T^{\Theta_0}, \tag{7}$$

where $\widetilde{\mathbb{E}}^{\Theta_0}$ denotes the expectation under $\widetilde{\mathbb{P}}_{\Theta_0}$, ξ^{Θ_0} is a predictable process for which we can determine the stochastic integral with respect to $\hat{S}$, and $\mathscr{L}^{\Theta_0}$ is a square integrable $\widetilde{\mathbb{P}}_{\Theta_0}$-martingale with $\mathscr{L}_0^{\Theta_0} = 0$, such that $\mathscr{L}^{\Theta_0}$ is $\widetilde{\mathbb{P}}_{\Theta_0}$-orthogonal to $\hat{S}$.

The quadratic hedging strategies are determined by the process ξ^{Θ_0}. It indicates the number of discounted risky assets to hold in the portfolio. The amount invested in the riskless asset is different in both strategies and is determined by the self-financing property for the MVH strategy and by the replicating condition for the RM strategy. See [19] for more details.

We define the process

$$\hat{V}_t^{\Theta_0} = \widetilde{\mathbb{E}}^{\Theta_0}[\hat{H}_T | \mathscr{F}_t], \quad \forall t \in [0, T],$$

which equals the value of the discounted portfolio for the RM strategy. The GKW-decomposition (7) implies that

$$\hat{V}_t^{\Theta_0} = \hat{V}_0^{\Theta_0} + \int_0^t \xi_s^{\Theta_0} \mathrm{d}\hat{S}_s + \mathscr{L}_t^{\Theta_0}, \quad \forall t \in [0, T]. \tag{8}$$

Moreover since $\mathscr{L}^{\Theta_0}$ is a $\widetilde{\mathbb{P}}_{\Theta_0}$-martingale, there exist processes X^{Θ_0} and $Y^{\Theta_0}(z)$ such that

$$\mathscr{L}_t^{\Theta_0} = \int_0^t X_s^{\Theta_0} \mathrm{d}W_s^{\Theta_0} + \int_0^t \int_{\mathbb{R}_0} Y_s^{\Theta_0}(z) \widetilde{N}^{\Theta_0}(\mathrm{d}s, \mathrm{d}z), \quad \forall t \in [0, T], \tag{9}$$

and which by the $\widetilde{\mathbb{P}}_{\Theta_0}$-orthogonality of $\mathscr{L}^{\Theta_0}$ and $\hat{S}$ satisfy

$$X^{\Theta_0} b + \int_{\mathbb{R}_0} Y^{\Theta_0}(z)(\mathrm{e}^z - 1)\rho(z; \Theta_0)\ell(\mathrm{d}z) = 0. \tag{10}$$

By substituting (6) and (9) in (8), we retrieve

$$\mathrm{d}\hat{V}_t^{\Theta_0} = \left(\xi_t^{\Theta_0} \hat{S}_t b + X_t^{\Theta_0}\right) \mathrm{d}W_t^{\Theta_0} + \int_{\mathbb{R}_0} \left(\xi_t^{\Theta_0} \hat{S}_t(\mathrm{e}^z - 1) + Y_t^{\Theta_0}(z)\right) \widetilde{N}^{\Theta_0}(\mathrm{d}t, \mathrm{d}z).$$

Let $\hat{\pi}^{\Theta_0} = \xi^{\Theta_0} \hat{S}$ indicate the amount of wealth invested in the discounted risky asset in a quadratic hedging strategy. We conclude that the following BSDEJ holds for the RM strategy

$$\begin{cases} \mathrm{d}\hat{V}_t^{\Theta_0} = A_t^{\Theta_0} \mathrm{d}W_t^{\Theta_0} + \int_{\mathbb{R}_0} B_t^{\Theta_0}(z) \widetilde{N}^{\Theta_0}(\mathrm{d}t, \mathrm{d}z), \\ \hat{V}_T^{\Theta_0} = \hat{H}_T, \end{cases} \tag{11}$$

where

$$A^{\Theta_0} = \hat{\pi}^{\Theta_0} b + X^{\Theta_0} \quad \text{and} \quad B^{\Theta_0}(z) = \hat{\pi}^{\Theta_0}(\mathrm{e}^z - 1) + Y^{\Theta_0}(z). \tag{12}$$

Since the random variable $\hat{H}_T$ is square integrable and $\mathscr{F}_T$-measurable, we know by [20] that the BSDEJ (11) has a unique solution $(\hat{V}^{\Theta_0}, A^{\Theta_0}, B^{\Theta_0})$. This follows from the fact that the drift parameter of $\hat{V}^{\Theta_0}$ equals zero under $\widetilde{\mathbb{P}}_{\Theta_0}$ and thus it is Lipschitz continuous.

We introduce another Lévy process L^ε, for $0 < \varepsilon < 1$, which is obtained by truncating the jumps of L with absolute size smaller than ε and replacing them by an independent Brownian motion which is appropriately scaled. The second stock price process is denoted by $S^\varepsilon = S_0 \mathrm{e}^{L^\varepsilon}$ and the corresponding discounted stock price process $\hat{S}^\varepsilon$ is thus given by

$$\mathrm{d}\hat{S}_t^\varepsilon = \hat{S}_t^\varepsilon \hat{a}_\varepsilon \mathrm{d}t + \hat{S}_t^\varepsilon b \mathrm{d}W_t + \hat{S}_t^\varepsilon \int_{\{|z|\geq\varepsilon\}} (\mathrm{e}^z - 1)\widetilde{N}(\mathrm{d}t, \mathrm{d}z) + \hat{S}_t^\varepsilon G(\varepsilon)\mathrm{d}\widetilde{W}_t, \tag{13}$$

for all $t \in [0, T]$ and $\hat{S}_0^\varepsilon = S_0$. Herein $\widetilde{W}$ is a standard Brownian motion independent of W,

$$G^2(\varepsilon) = \int_{\{|z|<\varepsilon\}} (\mathrm{e}^z - 1)^2 \ell(\mathrm{d}z), \text{ and} \tag{14}$$

$$\hat{a}_\varepsilon = a - r + \frac{1}{2}\left(b^2 + G^2(\varepsilon)\right) + \int_{\{|z|\geq\varepsilon\}} \left(\mathrm{e}^z - 1 - z1_{\{|z|<1\}}\right) \ell(\mathrm{d}z).$$

From now on, we assume that the filtration $\mathbb{F}$ is enlarged with the information of the Brownian motion $\widetilde{W}$ and we denote the new filtration by $\widetilde{\mathbb{F}}$. Moreover, we also assume absence of arbitrage in this second model. It is clear that the process L^ε has the Lévy characteristic triplet $\left(a, b^2 + G^2(\varepsilon), 1_{\{|\cdot|\geq\varepsilon\}}\ell\right)$ under the measure $\mathbb{P}$.

Let $\widetilde{\mathbb{P}}_\varepsilon$ represent a structure preserving martingale measure for $\hat{S}^\varepsilon$. The characteristic triplet of the driving process L^ε w.r.t. this martingale measure is denoted by $\left(\tilde{a}_\varepsilon, b^2 + G^2(\varepsilon), \tilde{\ell}_\varepsilon\right)$. From [14, Theorem III.3.24] we know that there exist a parameter $\Theta \in \mathbb{R}$ and a function $\rho(z; \Theta)$, $z \in \mathbb{R}$, under certain conditions, such that

$$\int_{\{\varepsilon\leq|z|<1\}} |z(\rho(z; \Theta) - 1)| \, \ell(\mathrm{d}z) < \infty, \tag{15}$$

$$\tilde{a}_\varepsilon = a + \left(b^2 + G^2(\varepsilon)\right) \Theta + \int_{\{\varepsilon\leq|z|<1\}} z\left(\rho(z; \Theta) - 1\right) \ell(\mathrm{d}z), \text{ and} \tag{16}$$

$$\tilde{\ell}_\varepsilon(\mathrm{d}z) = 1_{\{|z|\geq\varepsilon\}}\rho(z; \Theta)\ell(\mathrm{d}z). \tag{17}$$

Let us assume that Θ solves the following equation

$$\tilde{a}_\varepsilon - r + \frac{1}{2}\left(b^2 + G^2(\varepsilon)\right) + \int_{\mathbb{R}_0} \left(\mathrm{e}^z - 1 - z1_{\{|z|<1\}}\right) \tilde{\ell}_\varepsilon(\mathrm{d}z) = 0, \tag{18}$$

then $\hat{S}^\varepsilon$ is a martingale under $\widetilde{\mathbb{P}}$. From now on we indicate the solution of (18)—when it exists—as Θ_ε and the martingale measure as $\widetilde{\mathbb{P}}_{\Theta_\varepsilon}$.

The relation between the original measure $\mathbb{P}$ and the martingale measure $\widetilde{\mathbb{P}}_{\Theta_\varepsilon}$ is given by

$$\frac{\mathrm{d}\widetilde{\mathbb{P}}_{\Theta_\varepsilon}}{\mathrm{d}\mathbb{P}}\Big|_{\widetilde{\mathscr{F}}_t} = \exp\Bigg(b\Theta_\varepsilon W_t - \frac{1}{2}b^2{\Theta_0}^2 t + G(\varepsilon)\Theta_\varepsilon \widetilde{W}_t - \frac{1}{2}G^2(\varepsilon){\Theta_\varepsilon}^2 t$$
$$+ \int_0^t \int_{\{|z|\geq\varepsilon\}} \log(\rho(z;\Theta_\varepsilon))\widetilde{N}(\mathrm{d}s,\mathrm{d}z)$$
$$+ t\int_{\{|z|\geq\varepsilon\}} \big(\log(\rho(z;\Theta_\varepsilon)) + 1 - \rho(z;\Theta_\varepsilon)\big)\ell(\mathrm{d}z)\Bigg).$$

The processes W^{Θ_ε}, $\widetilde{W}^{\Theta_\varepsilon}$, and $\widetilde{N}^{\Theta_\varepsilon}$ defined by

$$\begin{aligned}
\mathrm{d}W_t^{\Theta_\varepsilon} &= \mathrm{d}W_t - b\Theta_\varepsilon \mathrm{d}t,\\
\mathrm{d}\widetilde{W}_t^{\Theta_\varepsilon} &= \mathrm{d}\widetilde{W}_t - G(\varepsilon)\Theta_\varepsilon \mathrm{d}t,\\
\widetilde{N}^{\Theta_\varepsilon}(\mathrm{d}t,\mathrm{d}z) &= N(\mathrm{d}t,\mathrm{d}z) - \rho(z;\Theta_\varepsilon)\ell(\mathrm{d}z)\mathrm{d}t\\
&= \widetilde{N}(\mathrm{d}t,\mathrm{d}z) + (1-\rho(z;\Theta_\varepsilon))\ell(\mathrm{d}z)\mathrm{d}t,
\end{aligned}\tag{19}$$

for all $t \in [0,T]$ and $z \in \{z \in \mathbb{R} : |z| \geq \varepsilon\}$, are two standard Brownian motions and a compensated jump measure under $\widetilde{\mathbb{P}}_{\Theta_\varepsilon}$ (see e.g. Theorem 1.33 in [17]). Hence the process $\hat{S}^\varepsilon$ is given by

$$\mathrm{d}\hat{S}_t^\varepsilon = \hat{S}_t^\varepsilon b\mathrm{d}W_t^{\Theta_\varepsilon} + \hat{S}_t^\varepsilon \int_{\{|z|\geq\varepsilon\}} (\mathrm{e}^z - 1)\widetilde{N}^{\Theta_\varepsilon}(\mathrm{d}t,\mathrm{d}z) + \hat{S}_t^\varepsilon G(\varepsilon)\mathrm{d}\widetilde{W}_t^{\Theta_\varepsilon}. \tag{20}$$

We consider an $\widetilde{\mathscr{F}}_T$-measurable and square integrable random variable H_T^ε which is the payoff of a contract. The discounted payoff is denoted by $\hat{H}_T^\varepsilon = \mathrm{e}^{-rT}H_T^\varepsilon$. The GKW-decomposition of $\hat{H}_T^\varepsilon$ under the martingale measure $\widetilde{\mathbb{P}}_{\Theta_\varepsilon}$ equals

$$\hat{H}_T^\varepsilon = \widetilde{\mathbb{E}}^{\Theta_\varepsilon}[\hat{H}_T^\varepsilon] + \int_0^T \xi_s^{\Theta_\varepsilon}\mathrm{d}\hat{S}_s^\varepsilon + \mathscr{L}_T^{\Theta_\varepsilon}, \tag{21}$$

where $\widetilde{\mathbb{E}}^{\Theta_\varepsilon}$ is the expectation under $\widetilde{\mathbb{P}}_{\Theta_\varepsilon}$, ξ^{Θ_ε} is a predictable process for which we can determine the stochastic integral with respect to $\hat{S}^\varepsilon$, and $\mathscr{L}^{\Theta_\varepsilon}$ is a square integrable $\widetilde{\mathbb{P}}_{\Theta_\varepsilon}$-martingale with $\mathscr{L}_0^{\Theta_\varepsilon} = 0$, such that $\mathscr{L}^{\Theta_\varepsilon}$ is $\widetilde{\mathbb{P}}_{\Theta_\varepsilon}$-orthogonal to $\hat{S}^\varepsilon$.

The value of the discounted portfolio for the RM strategy is defined by

$$\hat{V}_t^{\Theta_\varepsilon} = \widetilde{\mathbb{E}}^{\Theta_\varepsilon}[\hat{H}_T^\varepsilon|\widetilde{\mathscr{F}}_t], \quad \forall t \in [0,T].$$

From the GKW-decomposition (21) we have

$$\hat{V}_t^{\Theta_\varepsilon} = \hat{V}_0^{\Theta_\varepsilon} + \int_0^t \xi_s^{\Theta_\varepsilon}\mathrm{d}\hat{S}_s^\varepsilon + \mathscr{L}_t^{\Theta_\varepsilon}, \quad \forall t \in [0,T]. \tag{22}$$

Moreover since $\mathscr{L}^{\Theta_\varepsilon}$ is a $\widetilde{\mathbb{P}}_{\Theta_\varepsilon}$-martingale, there exist processes X^{Θ_ε}, $Y^{\Theta_\varepsilon}(z)$, and Z^{Θ_ε} such that

$$\mathscr{L}_t^{\Theta_\varepsilon} = \int_0^t X_s^{\Theta_\varepsilon} \mathrm{d}W_s^{\Theta_\varepsilon} + \int_0^t \int_{\{|z|\geq\varepsilon\}} Y_s^{\Theta_\varepsilon}(z) \widetilde{N}^{\Theta_\varepsilon}(\mathrm{d}s, \mathrm{d}z) + \int_0^t Z_s^{\Theta_\varepsilon} \mathrm{d}\widetilde{W}_s^{\Theta_\varepsilon}, \quad \forall t \in [0, T]. \tag{23}$$

The $\widetilde{\mathbb{P}}_{\Theta_\varepsilon}$-orthogonality of $\mathscr{L}^{\Theta_\varepsilon}$ and $\hat{S}^\varepsilon$ implies that

$$X^{\Theta_\varepsilon} b + \int_{\{|z|\geq\varepsilon\}} Y^{\Theta_\varepsilon}(z)(\mathrm{e}^z - 1)\rho(z; \Theta_\varepsilon)\ell(\mathrm{d}z) + Z^{\Theta_\varepsilon} G(\varepsilon) = 0. \tag{24}$$

Combining (20) and (23) in (22), we get

$$\begin{aligned} \mathrm{d}\hat{V}_t^{\Theta_\varepsilon} &= \left(\xi_t^{\Theta_\varepsilon} \hat{S}_t^\varepsilon b + X_t^{\Theta_\varepsilon}\right) \mathrm{d}W_t^{\Theta_\varepsilon} + \int_{\{|z|\geq\varepsilon\}} \left(\xi_t^{\Theta_\varepsilon} \hat{S}_t^\varepsilon (\mathrm{e}^z - 1) + Y_t^{\Theta_\varepsilon}(z)\right) \widetilde{N}^{\Theta_\varepsilon}(\mathrm{d}t, \mathrm{d}z) \\ &\quad + \left(\xi_t^{\Theta_\varepsilon} \hat{S}_t^\varepsilon G(\varepsilon) + Z_t^{\Theta_\varepsilon}\right) \mathrm{d}\widetilde{W}_t^{\Theta_\varepsilon}. \end{aligned}$$

Let $\hat{\pi}^{\Theta_\varepsilon} = \xi^{\Theta_\varepsilon} \hat{S}^\varepsilon$ denote the amount of wealth invested in the discounted risky asset in the quadratic hedging strategy. We conclude that the following BSDEJ holds for the RM strategy

$$\begin{cases} \mathrm{d}\hat{V}_t^{\Theta_\varepsilon} = A_t^{\Theta_\varepsilon} \mathrm{d}W_t^{\Theta_\varepsilon} + \int_{\{|z|\geq\varepsilon\}} B_t^{\Theta_\varepsilon}(z) \widetilde{N}^{\Theta_\varepsilon}(\mathrm{d}t, \mathrm{d}z) + C_t^{\Theta_\varepsilon} \mathrm{d}\widetilde{W}_t^{\Theta_\varepsilon}, \\ \hat{V}_T^{\Theta_\varepsilon} = \hat{H}_T^\varepsilon, \end{cases} \tag{25}$$

where

$$\begin{aligned} &A^{\Theta_\varepsilon} = \hat{\pi}^{\Theta_\varepsilon} b + X^{\Theta_\varepsilon}, \quad B^{\Theta_\varepsilon}(z) = \hat{\pi}^{\Theta_\varepsilon}(\mathrm{e}^z - 1) + Y^{\Theta_\varepsilon}(z), \quad \text{and} \\ &C^{\Theta_\varepsilon} = \hat{\pi}^{\Theta_\varepsilon} G(\varepsilon) + Z^{\Theta_\varepsilon}. \end{aligned} \tag{26}$$

Since the random variable $\hat{H}_T^\varepsilon$ is square integrable and $\widetilde{\mathscr{F}}_T$-measurable we know by [20] that the BSDEJ (25) has a unique solution $\left(\hat{V}^{\Theta_\varepsilon}, A^{\Theta_\varepsilon}, B^{\Theta_\varepsilon}, C^{\Theta_\varepsilon}\right)$. This results from the fact that the drift parameter of $\hat{V}^{\Theta_\varepsilon}$ equals zero under $\widetilde{\mathbb{P}}_{\Theta_\varepsilon}$ and thus is Lipschitz continuous.

3 Robustness of the Quadratic Hedging Strategies

The aim of this section is to study the stability of the quadratic hedging strategies to the variation of the model, where we consider the two stock price models defined in (1) and (13). We study the stability of the RM strategy extensively and at the end of this section we come back to the MVH strategy. Since we work in the martingale setting, we first present some specific martingale measures which are commonly used in finance and in electricity markets. Then we discuss some common properties which are fulfilled by these measures. This is the topic of the next subsection.

3.1 Robustness of the Martingale Measures

Recall from the previous section that the martingale measures $\widetilde{\mathbb{P}}_{\Theta_0}$ and $\widetilde{\mathbb{P}}_{\Theta_\varepsilon}$ are determined via the functions $\rho(.\,;\Theta_0)$, $\rho(.\,;\Theta_\varepsilon)$ and the parameters Θ_0, Θ_ε, respectively. We present the following assumptions on these characteristics.

Assumptions 1 For Θ_0, Θ_ε, $\rho(.\,;\Theta_0)$, and $\rho(.\,;\Theta_\varepsilon)$ satisfying Eqs. (2)–(4), and Eqs. (15)–(18) we assume the following, where C denotes a positive constant and $\Theta \in \{\Theta_0, \Theta_\varepsilon\}$.

(i) Θ_0 and Θ_ε exist and are unique.

(ii) It holds that

$$|\Theta_0 - \Theta_\varepsilon| \le C\widetilde{G}^2(\varepsilon),$$

where $\widetilde{G}(\varepsilon) = \max(G(\varepsilon), \sigma(\varepsilon))$. Herein $\sigma(\varepsilon)$ equals the standard deviation of the jumps of L with size smaller than ε, i.e.

$$\sigma^2(\varepsilon) = \int_{\{|z|<\varepsilon\}} z^2 \ell(\mathrm{d}z).$$

(iii) On the other hand, Θ_ε is uniformly bounded in ε, i.e.

$$|\Theta_\varepsilon| \le C.$$

(iv) For all z in $\{|z| < 1\}$ it holds that

$$|\rho(z;\Theta)| \le C.$$

(v) We have

$$\int_{\{|z|\ge 1\}} \rho^4(z;\Theta)\ell(\mathrm{d}z) \le C.$$

(vi) It is guaranteed that

$$\int_{\mathbb{R}_0} \big(1 - \rho(z;\Theta)\big)^2 \ell(\mathrm{d}z) \le C.$$

(vii) It holds for $k \in \{2, 4\}$ that

$$\int_{\mathbb{R}_0} \big(\rho(z; \Theta_0) - \rho(z; \Theta_\varepsilon)\big)^k \ell(\mathrm{d}z) \le C\widetilde{G}^{2k}(\varepsilon).$$

Widely used martingale measures in the exponential Lévy setting are the Esscher transform (ET), minimal entropy martingale measure (MEMM), and minimal martingale measure (MMM), which are specified as follows.

- In order to define the ET we assume that

$$\int_{\{|z|\ge 1\}} \mathrm{e}^{\theta z} \ell(\mathrm{d}z) < \infty, \qquad \forall \theta \in \mathbb{R}. \tag{27}$$

  The Lévy measures under the ET are given in (3) and (17) where $\rho(z; \Theta) = \mathrm{e}^{\Theta z}$. The ET for the first model is then determined by the parameter Θ_0 satisfying (4). For the second model the ET corresponds to the solution Θ_ε of (18). See [13] for more details.
- Let us impose that

$$\int_{\{|z|\ge 1\}} \mathrm{e}^{\theta(\mathrm{e}^z - 1)} \ell(\mathrm{d}z) < \infty, \qquad \forall \theta \in \mathbb{R}, \tag{28}$$

  and that $\rho(z; \Theta) = \mathrm{e}^{\Theta(\mathrm{e}^z - 1)}$ in the Lévy measures. Then the solution Θ_0 of Eq. (4) determines the MEMM for the first model, and Θ_ε being the solution of (18) characterises the MEMM for the second model. The MEMM is studied in [12].
- Let us consider the assumption

$$\int_{\{z\ge 1\}} \mathrm{e}^{4z} \ell(\mathrm{d}z) < \infty. \tag{29}$$

  The MMM implies that $\rho(z; \Theta) = \Theta(\mathrm{e}^z - 1) - 1$ in the Lévy measures and the parameters Θ_0 and Θ_ε are the solutions of (4) and (18). More information about the MMM can be found in [1, 10].

In [4, 6] it was shown that the ET, the MEMM, and the MMM fulfill statements (i), (ii), (iii), and (iv) of Assumptions 1 in the exponential Lévy setting. The following proposition shows that items (v), (vi), and (vii) of Assumptions 1 also hold for these martingale measures.

Proposition 1 *The Lévy measures given in* (3) *and* (17) *and corresponding to the ET, MEMM, and MMM, satisfy (v), (vi), and (vii) of Assumptions* 1.

Proof Recall that the Lévy measure satisfies the following integrability conditions

$$\int_{\{|z|<1\}} z^2 \ell(\mathrm{d}z) < \infty \qquad \text{and} \qquad \int_{\{|z|\geq 1\}} \ell(\mathrm{d}z) < \infty. \tag{30}$$

We show that the statement holds for the considered martingale measures.

- Under the ET it holds for $\Theta \in \{\Theta_0, \Theta_\varepsilon\}$ that

$$\rho^4(z;\Theta) = \mathrm{e}^{4\Theta z} \leq \mathrm{e}^{4C|z|},$$

because of (iii) in Assumptions 1. By the mean value theorem (MVT), there exists a number Θ' between 0 and Θ such that

$$\big(1-\rho(z;\Theta)\big)^2 = z^2 \mathrm{e}^{2\Theta' z}\Theta^2 \leq \Big(1_{\{|z|<1\}}\mathrm{e}^{2C}z^2 + 1_{\{|z|\geq 1\}}\mathrm{e}^{(2C+2)z}\Big)\, C,$$

where we used again Assumptions 1 (iii). For $k \in \{2,4\}$, we derive via the MVT that

$$\big(\rho(z;\Theta_0)-\rho(z;\Theta_\varepsilon)\big)^k = \mathrm{e}^{k\Theta_0 z}\Big(1-\mathrm{e}^{(\Theta_\varepsilon-\Theta_0)z}\Big)^k = \mathrm{e}^{k\Theta_0 z} z^k \mathrm{e}^{k\Theta'' z}(\Theta_0-\Theta_\varepsilon)^k,$$

where Θ'' is a number between 0 and $\Theta_\varepsilon - \Theta_0$. Assumptions 1 (ii) imply that

$$\big(\rho(z;\Theta_0)-\rho(z;\Theta_\varepsilon)\big)^k \leq \Big(1_{\{|z|<1\}}\mathrm{e}^{k(|\Theta_0|+C)}z^2 + 1_{\{|z|\geq 1\}}\mathrm{e}^{k(\Theta_0+1+C)z}\Big)\, C\widetilde{G}^{2k}(\varepsilon).$$

The obtained inequalities and integrability conditions (27) and (30) prove the statement.

- Consider the MEMM and $\Theta \in \{\Theta_0, \Theta_\varepsilon\}$. We have

$$\rho^4(z;\Theta) = \mathrm{e}^{4\Theta(\mathrm{e}^z-1)} \leq \mathrm{e}^{4C|\mathrm{e}^z-1|},$$

because of (iii) in Assumptions 1. The latter assumption and the MVT imply that

$$\begin{aligned}\big(1-\rho(z;\Theta)\big)^2 &= (\mathrm{e}^z-1)^2\mathrm{e}^{2\Theta'(\mathrm{e}^z-1)}\Theta^2\\ &\leq \Big(1_{\{|z|<1\}}\mathrm{e}^{2C(\mathrm{e}+1)+2}z^2 + 1_{\{|z|\geq 1\}}\mathrm{e}^{(2C+2)(\mathrm{e}^z-1)}\Big)\, C.\end{aligned}$$

We determine via the MVT and properties (ii) and (iii) in Assumptions 1 for $k \in \{2,4\}$ that

$$\begin{aligned}&\big(\rho(z;\Theta_0)-\rho(z;\Theta_\varepsilon)\big)^k\\ &= \mathrm{e}^{k\Theta_0(\mathrm{e}^z-1)}\Big(1-\mathrm{e}^{(\Theta_\varepsilon-\Theta_0)(\mathrm{e}^z-1)}\Big)^k\\ &= \mathrm{e}^{k\Theta_0(\mathrm{e}^z-1)}(\mathrm{e}^z-1)^k\mathrm{e}^{k\Theta''(\mathrm{e}^z-1)}(\Theta_0-\Theta_\varepsilon)^k\\ &\leq \Big(1_{\{|z|<1\}}\mathrm{e}^{k(|\Theta_0|(\mathrm{e}+1)+1+C(\mathrm{e}+1))}z^2 + 1_{\{|z|\geq 1\}}\mathrm{e}^{k(\Theta_0+1+C)(\mathrm{e}^z-1)}\Big)\, C\widetilde{G}^{2k}(\varepsilon).\end{aligned}$$

From (28) and (30) we conclude that (v), (vi), and (vii) in Assumptions 1 are in force.

- For the MMM we have

$$\rho^4(z;\Theta) = \big(\Theta(\mathrm{e}^z - 1) - 1\big)^4 \le C(\mathrm{e}^{4z} + 1).$$

Moreover it holds that

$$\big(1 - \rho(z;\Theta)\big)^2 = (\mathrm{e}^z - 1)^2\Theta^2 \le \Big(1_{\{|z|<1\}}\mathrm{e}^2 z^2 + 1_{\{|z|\ge 1\}}(\mathrm{e}^{2z} + 1)\Big)\, C.$$

We get through (ii) and (iii) in Assumptions 1 that

$$\begin{aligned}\big(\rho(z;\Theta_0) - \rho(z;\Theta_\varepsilon)\big)^k &= (\mathrm{e}^z - 1)^k(\Theta_0 - \Theta_\varepsilon)^k \\ &\le \Big(1_{\{|z|<1\}}\mathrm{e}^k z^2 + 1_{\{|z|\ge 1\}}(\mathrm{e}^{kz} + 1)\Big)\, C\widetilde{G}^{2k}(\varepsilon),\end{aligned}$$

for $k \in \{2, 4\}$. The proof is completed by involving conditions (29) and (30). □

3.2 Robustness of the BSDEJ

The aim of this subsection is to study the robustness of the BSDEJs (11) and (25). First, we prove the L^2-boundedness of the solution of the BSDEJ (11) in the following lemma.

Lemma 1 *Assume point (vi) from Assumptions 1. Let $(\hat{V}^{\Theta_0}, A^{\Theta_0}, B^{\Theta_0})$ be the solution of (11). Then we have for all $t \in [0, T]$*

$$\mathbb{E}\left[\int_t^T (\hat{V}_s^{\Theta_0})^2 \mathrm{d}s\right] + \mathbb{E}\left[\int_t^T (A_s^{\Theta_0})^2 \mathrm{d}s\right] + \mathbb{E}\left[\int_t^T \int_{\mathbb{R}_0} (B_s^{\Theta_0}(z))^2 \ell(\mathrm{d}z)\mathrm{d}s\right] \le C\mathbb{E}[\hat{H}_T^2],$$

where C represents a positive constant.

Proof Via (5) we rewrite the BSDEJ (11) as follows

$$\begin{aligned}\mathrm{d}\hat{V}_t^{\Theta_0} &= \left(-b\Theta_0 A_t^{\Theta_0} + \int_{\mathbb{R}_0} B_t^{\Theta_0}(z)(1 - \rho(z;\Theta_0))\ell(\mathrm{d}z)\right)\mathrm{d}t \\ &\quad + A_t^{\Theta_0}\mathrm{d}W_t + \int_{\mathbb{R}_0} B_t^{\Theta_0}(z)\widetilde{N}(\mathrm{d}t, \mathrm{d}z).\end{aligned}$$

We apply the Itô formula to $\mathrm{e}^{\beta t}(\hat{V}_t^{\Theta_0})^2$ and find that

$$
\begin{aligned}
&\mathrm{d}\left(\mathrm{e}^{\beta t}(\hat{V}_t^{\Theta_0})^2\right)\\
&= \beta\mathrm{e}^{\beta t}(\hat{V}_t^{\Theta_0})^2\mathrm{d}t + 2\mathrm{e}^{\beta t}\hat{V}_t^{\Theta_0}\left(-b\Theta_0 A_t^{\Theta_0} + \int_{\mathbb{R}_0} B_t^{\Theta_0}(z)(1-\rho(z;\Theta_0))\ell(\mathrm{d}z)\right)\mathrm{d}t\\
&\quad + 2\mathrm{e}^{\beta t}\hat{V}_t^{\Theta_0}A_t^{\Theta_0}\mathrm{d}W_t + \mathrm{e}^{\beta t}(A_t^{\Theta_0})^2\mathrm{d}t\\
&\quad + \int_{\mathbb{R}_0}\mathrm{e}^{\beta t}\left(\left(\hat{V}_{t-}^{\Theta_0} + B_t^{\Theta_0}(z)\right)^2 - (\hat{V}_{t-}^{\Theta_0})^2\right)\widetilde{N}(\mathrm{d}t,\mathrm{d}z) + \int_{\mathbb{R}_0}\mathrm{e}^{\beta t}(B_t^{\Theta_0}(z))^2\ell(\mathrm{d}z)\mathrm{d}t.
\end{aligned}
$$

By integration and taking the expectation we recover that

$$
\begin{aligned}
&\mathbb{E}\left[\mathrm{e}^{\beta t}(\hat{V}_t^{\Theta_0})^2\right]\\
&= \mathbb{E}\left[\mathrm{e}^{\beta T}(\hat{V}_T^{\Theta_0})^2\right] - \beta\mathbb{E}\left[\int_t^T \mathrm{e}^{\beta s}(\hat{V}_s^{\Theta_0})^2\mathrm{d}s\right]\\
&\quad - 2\mathbb{E}\left[\int_t^T \mathrm{e}^{\beta s}\hat{V}_s^{\Theta_0}\left(-b\Theta_0 A_s^{\Theta_0} + \int_{\mathbb{R}_0} B_s^{\Theta_0}(z)(1-\rho(z;\Theta_0))\ell(\mathrm{d}z)\right)\mathrm{d}s\right] \qquad (31)\\
&\quad - \mathbb{E}\left[\int_t^T \mathrm{e}^{\beta s}(A_s^{\Theta_0})^2\mathrm{d}s\right] - \mathbb{E}\left[\int_t^T\int_{\mathbb{R}_0}\mathrm{e}^{\beta s}(B_s^{\Theta_0}(z))^2\ell(\mathrm{d}z)\mathrm{d}s\right].
\end{aligned}
$$

Because of the properties

$$
\text{for all } a, b \in \mathbb{R} \text{ and } k \in \mathbb{R}_0^+ \text{ it holds that } \pm 2ab \le ka^2 + \frac{1}{k}b^2 \tag{32}
$$

and

$$
\text{for all } n \in \mathbb{N} \text{ and for all } a_i \in \mathbb{R}, i = 1, \ldots, n \text{ we have that } \left(\sum_{i=1}^n a_i\right)^2 \le n\sum_{i=1}^n a_i^2, \tag{33}
$$

the third term in the right hand side of (31) is estimated by

$$
\begin{aligned}
&- 2\mathbb{E}\left[\int_t^T \mathrm{e}^{\beta s}\hat{V}_s^{\Theta_0}\left(-b\Theta_0 A_s^{\Theta_0} + \int_{\mathbb{R}_0} B_s^{\Theta_0}(z)(1-\rho(z;\Theta_0))\ell(\mathrm{d}z)\right)\mathrm{d}s\right]\\
&\le \mathbb{E}\left[\int_t^T \mathrm{e}^{\beta s}\left\{k(\hat{V}_s^{\Theta_0})^2 + \frac{1}{k}\left(-b\Theta_0 A_s^{\Theta_0} + \int_{\mathbb{R}_0} B_s^{\Theta_0}(z)(1-\rho(z;\Theta_0))\ell(\mathrm{d}z)\right)^2\right\}\mathrm{d}s\right]\\
&\le k\mathbb{E}\left[\int_t^T \mathrm{e}^{\beta s}(\hat{V}_s^{\Theta_0})^2\mathrm{d}s\right] + \frac{2}{k}b^2{\Theta_0}^2\mathbb{E}\left[\int_t^T \mathrm{e}^{\beta s}(A_s^{\Theta_0})^2\mathrm{d}s\right]\\
&\quad + \frac{2}{k}\int_{\mathbb{R}_0}(1-\rho(z;\Theta_0))^2\,\ell(\mathrm{d}z)\mathbb{E}\left[\int_t^T \mathrm{e}^{\beta s}\int_{\mathbb{R}_0}(B_s^{\Theta_0}(z))^2\ell(\mathrm{d}z)\mathrm{d}s\right].
\end{aligned}
$$

Substituting the latter inequality in (31) leads to

$$\mathbb{E}\left[e^{\beta t}(\hat{V}_t^{\Theta_0})^2\right]+(\beta-k)\mathbb{E}\left[\int_t^T e^{\beta s}(\hat{V}_s^{\Theta_0})^2 ds\right]+\left(1-\frac{2}{k}b^2{\Theta_0}^2\right)\mathbb{E}\left[\int_t^T e^{\beta s}(A_s^{\Theta_0})^2 ds\right]$$
$$+\left(1-\frac{2}{k}\int_{\mathbb{R}_0}(1-\rho(z;\Theta_0))^2\,\ell(dz)\right)\mathbb{E}\left[\int_t^T e^{\beta s}\int_{\mathbb{R}_0}(B_s^{\Theta_0}(z))^2\ell(dz)ds\right]$$
$$\leq \mathbb{E}\left[e^{\beta T}(\hat{V}_T^{\Theta_0})^2\right]. \tag{34}$$

Let k guarantee that

$$1-\frac{2}{k}b^2{\Theta_0}^2\geq\frac{1}{2}\quad\text{and}\quad 1-\frac{2}{k}\int_{\mathbb{R}_0}(1-\rho(z;\Theta_0))^2\,\ell(dz)\geq\frac{1}{2}.$$

Hence we choose

$$k\geq 4\max\left(b^2{\Theta_0}^2,\int_{\mathbb{R}_0}(1-\rho(z;\Theta_0))^2\,\ell(dz)\right)>0,$$

which exists because of (vi) from Assumptions 1. Besides we assume that $\beta\geq k+\frac{1}{2}>0$. Then for $s\in[0,T]$ it follows that $1\leq e^{\beta s}\leq e^{\beta T}$ and from (34) we achieve

$$\mathbb{E}\left[\int_t^T(\hat{V}_s^{\Theta_0})^2 ds\right]+\mathbb{E}\left[\int_t^T(A_s^{\Theta_0})^2 ds\right]+\mathbb{E}\left[\int_t^T\int_{\mathbb{R}_0}(B_s^{\Theta_0}(z))^2\ell(dz)ds\right]\leq C\mathbb{E}[(\hat{V}_T^{\Theta_0})^2],$$

which proves the claim. □

In order to study the robustness of the BSDEJs (11) and (25), we consider both models under the enlarged filtration $\widetilde{\mathbb{F}}$ since we have for all $t\in[0,T]$ that $\mathscr{F}_t\subset\widetilde{\mathscr{F}}_t$. Let us define

$$\bar{V}^{\varepsilon}=\hat{V}^{\Theta_0}-\hat{V}^{\Theta_\varepsilon},\quad \bar{A}^{\varepsilon}=A^{\Theta_0}-A^{\Theta_\varepsilon},\quad \bar{B}^{\varepsilon}(z)=B^{\Theta_0}(z)-1_{\{|z|\geq\varepsilon\}}B^{\Theta_\varepsilon}(z).$$

We derive from (5), (11), (19), and (25) that

$$d\bar{V}_t^{\varepsilon}=\alpha_t^{\varepsilon}dt+\bar{A}_t^{\varepsilon}dW_t+\int_{\mathbb{R}_0}\bar{B}_t^{\varepsilon}(z)\widetilde{N}(dt,dz)-C_t^{\Theta_\varepsilon}d\widetilde{W}_t, \tag{35}$$

where

$$\alpha^{\varepsilon}=-b(\Theta_0A^{\Theta_0}-\Theta_\varepsilon A^{\Theta_\varepsilon})+G(\varepsilon)\Theta_\varepsilon C^{\Theta_\varepsilon} \tag{36}$$
$$+\int_{\mathbb{R}_0}\left(B^{\Theta_0}(z)\,(1-\rho(z;\Theta_0))-1_{\{|z|\geq\varepsilon\}}B^{\Theta_\varepsilon}(z)\,(1-\rho(z;\Theta_\varepsilon))\right)\ell(dz).$$

The process α^ε (36) plays a crucial role in the study of the robustness of the BSDEJ. In the following lemma we state an upper bound for this process in terms of the solutions of the BSDEJs.

Lemma 2 *Let Assumptions 1 hold true. Consider α^ε as defined in (36). For any $t \in [0, T]$ and $\beta \in \mathbb{R}$ it holds that*

$$\begin{aligned}
&\mathbb{E}\left[\int_t^T \mathrm{e}^{\beta s}(\alpha_s^\varepsilon)^2 \mathrm{d}s\right] \\
&\leq C\Bigg(\widetilde{G}^4(\varepsilon)\left\{\mathbb{E}\left[\int_t^T \mathrm{e}^{\beta s}(A_s^{\Theta_0})^2 \mathrm{d}s\right] + \mathbb{E}\left[\int_t^T \mathrm{e}^{\beta s}\int_{\mathbb{R}_0}(B_s^{\Theta_0}(z))^2 \ell(\mathrm{d}z)\mathrm{d}s\right]\right\} \\
&\qquad + \mathbb{E}\left[\int_t^T \mathrm{e}^{\beta s}(\bar{A}_s^\varepsilon)^2 \mathrm{d}s\right] + \mathbb{E}\left[\int_t^T \mathrm{e}^{\beta s}\int_{\mathbb{R}_0}(\bar{B}_s^\varepsilon(z))^2 \ell(\mathrm{d}z)\mathrm{d}s\right] \\
&\qquad + \mathbb{E}\left[\int_t^T \mathrm{e}^{\beta s}(C_s^{\Theta_\varepsilon})^2 \mathrm{d}s\right]\Bigg),
\end{aligned}$$

where C is a positive constant.

Proof Parts (ii) and (iii) of Assumptions 1 imply that

$$\begin{aligned}
\left|-b\left(\Theta_0 A_s^{\Theta_0} - \Theta_\varepsilon A_s^{\Theta_\varepsilon}\right)\right| &\leq |b||\Theta_0 - \Theta_\varepsilon||A_s^{\Theta_0}| + |b||\Theta_\varepsilon||A_s^{\Theta_0} - A_s^{\Theta_\varepsilon}| \\
&\leq C\widetilde{G}^2(\varepsilon)|A_s^{\Theta_0}| + C|\bar{A}_s^\varepsilon|
\end{aligned}$$

and

$$|G(\varepsilon)\Theta_\varepsilon C_s^{\Theta_\varepsilon}| \leq C|C_s^{\Theta_\varepsilon}|.$$

From Hölder's inequality and Assumptions 1 (vi) and (vii) it follows that

$$\begin{aligned}
&\left|\int_{\mathbb{R}_0}\left(B_s^{\Theta_0}(z)\,(1-\rho(z;\Theta_0)) - 1_{\{|z|\geq\varepsilon\}}B_s^{\Theta_\varepsilon}(z)\,(1-\rho(z;\Theta_\varepsilon))\right)\ell(\mathrm{d}z)\right| \\
&\leq \left|\int_{\mathbb{R}_0} B_s^{\Theta_0}(z)\,(\rho(z;\Theta_0)-\rho(z;\Theta_\varepsilon))\,\ell(\mathrm{d}z)\right| + \left|\int_{\mathbb{R}_0}\bar{B}_s^\varepsilon(z)\,(1-\rho(z;\Theta_\varepsilon))\,\ell(\mathrm{d}z)\right| \\
&\leq \left(\int_{\mathbb{R}_0}\left(\rho(z;\Theta_0)-\rho(z;\Theta_\varepsilon)\right)^2\ell(\mathrm{d}z)\right)^{1/2}\left(\int_{\mathbb{R}_0}(B_s^{\Theta_0}(z))^2\ell(\mathrm{d}z)\right)^{1/2} \\
&\quad + \left(\int_{\mathbb{R}_0}\left(1-\rho(z;\Theta_\varepsilon)\right)^2\ell(\mathrm{d}z)\right)^{1/2}\left(\int_{\mathbb{R}_0}(\bar{B}_s^\varepsilon(z))^2\ell(\mathrm{d}z)\right)^{1/2} \\
&\leq C\widetilde{G}^2(\varepsilon)\left(\int_{\mathbb{R}_0}(B_s^{\Theta_0}(z))^2\ell(\mathrm{d}z)\right)^{1/2} + C\left(\int_{\mathbb{R}_0}(\bar{B}_s^\varepsilon(z))^2\ell(\mathrm{d}z)\right)^{1/2}.
\end{aligned}$$

We conclude that

$$(\alpha_s^\varepsilon)^2 \le C\left(\widetilde{G}^4(\varepsilon)\left\{(A_s^{\Theta_0})^2 + \int_{\mathbb{R}_0}(B_s^{\Theta_0}(z))^2\ell(\mathrm{d}z)\right\}\right.$$
$$\left. + (\bar{A}_s^\varepsilon)^2 + \int_{\mathbb{R}_0}(\bar{B}_s^\varepsilon(z))^2\ell(\mathrm{d}z) + (C_s^{\Theta_\varepsilon})^2\right).$$

The statement is easily deduced from this inequality. □

With these two lemmas ready for use, we state and prove the main result of this subsection which is the robustness of the BSDEJs for the discounted portfolio value process of the RM strategy.

Theorem 1 *Assumptions 1 are in force. Let* $(\hat{V}^{\Theta_0}, A^{\Theta_0}, B^{\Theta_0})$ *be the solution of* (11) *and* $(\hat{V}^{\Theta_\varepsilon}, A^{\Theta_\varepsilon}, B^{\Theta_\varepsilon}, C^{\Theta_\varepsilon})$ *be the solution of* (25)*. For some positive constant C and any* $t \in [0, T]$ *we have*

$$\mathbb{E}\left[\int_t^T (\hat{V}_s^{\Theta_0} - \hat{V}_s^{\Theta_\varepsilon})^2\mathrm{d}s\right] + \mathbb{E}\left[\int_t^T (A_s^{\Theta_0} - A_s^{\Theta_\varepsilon})^2\mathrm{d}s\right]$$
$$+ \mathbb{E}\left[\int_t^T\int_{\mathbb{R}_0}\left(B_s^{\Theta_0}(z) - 1_{\{|z|\ge\varepsilon\}}B_s^{\Theta_\varepsilon}(z)\right)^2\ell(\mathrm{d}z)\mathrm{d}s\right] + \mathbb{E}\left[\int_t^T (C_s^{\Theta_\varepsilon})^2\mathrm{d}s\right]$$
$$\le C\left(\mathbb{E}\left[(\hat{H}_T - \hat{H}_T^\varepsilon)^2\right] + \widetilde{G}^4(\varepsilon)\mathbb{E}[\hat{H}_T^2]\right).$$

Proof We apply the Itô formula to $\mathrm{e}^{\beta t}(\bar{V}_t^\varepsilon)^2$

$$\mathrm{d}\left(\mathrm{e}^{\beta t}(\bar{V}_t^\varepsilon)^2\right) = \beta\mathrm{e}^{\beta t}(\bar{V}_t^\varepsilon)^2\mathrm{d}t + 2\mathrm{e}^{\beta t}\bar{V}_t^\varepsilon\alpha_t^\varepsilon\mathrm{d}t + 2\mathrm{e}^{\beta t}\bar{V}_t^\varepsilon\bar{A}_t^\varepsilon\mathrm{d}W_t - 2\mathrm{e}^{\beta t}\bar{V}_t^\varepsilon C_t^{\Theta_\varepsilon}\mathrm{d}\widetilde{W}_t$$
$$+ \mathrm{e}^{\beta t}(\bar{A}_t^\varepsilon)^2\mathrm{d}t + \mathrm{e}^{\beta t}(C_t^{\Theta_\varepsilon})^2\mathrm{d}t + \int_{\mathbb{R}_0}\mathrm{e}^{\beta t}(\bar{B}_t^\varepsilon(z))^2\ell(\mathrm{d}z)\mathrm{d}t$$
$$+ \int_{\mathbb{R}_0}\mathrm{e}^{\beta t}\left((\bar{V}_{t-}^\varepsilon + \bar{B}_t^\varepsilon(z))^2 - (\bar{V}_{t-}^\varepsilon)^2\right)\widetilde{N}(\mathrm{d}t, \mathrm{d}z).$$

Integration over the interval $[t, T]$ and taking the expectation under $\mathbb{P}$ results into

$$\mathbb{E}\left[\mathrm{e}^{\beta t}(\bar{V}_t^\varepsilon)^2\right] = \mathbb{E}\left[\mathrm{e}^{\beta T}(\bar{V}_T^\varepsilon)^2\right] - \beta\mathbb{E}\left[\int_t^T \mathrm{e}^{\beta s}(\bar{V}_s^\varepsilon)^2\mathrm{d}s\right] - 2\mathbb{E}\left[\int_t^T \mathrm{e}^{\beta s}\bar{V}_s^\varepsilon\alpha_s^\varepsilon\mathrm{d}s\right]$$
$$- \mathbb{E}\left[\int_t^T \mathrm{e}^{\beta s}(\bar{A}_s^\varepsilon)^2\mathrm{d}s\right] - \mathbb{E}\left[\int_t^T\int_{\mathbb{R}_0}\mathrm{e}^{\beta s}(\bar{B}_s^\varepsilon(z))^2\ell(\mathrm{d}z)\mathrm{d}s\right]$$
$$- \mathbb{E}\left[\int_t^T \mathrm{e}^{\beta s}(C_s^{\Theta_\varepsilon})^2\mathrm{d}s\right],$$

or equivalently

$$
\begin{aligned}
&\mathbb{E}\left[\mathrm{e}^{\beta t}(\bar{V}_t^{\varepsilon})^2\right] + \mathbb{E}\left[\int_t^T \mathrm{e}^{\beta s}(\bar{A}_s^{\varepsilon})^2 \mathrm{d}s\right] \\
&\quad + \mathbb{E}\left[\int_t^T \int_{\mathbb{R}_0} \mathrm{e}^{\beta s}(\bar{B}_s^{\varepsilon}(z))^2 \ell(\mathrm{d}z)\mathrm{d}s\right] + \mathbb{E}\left[\int_t^T \mathrm{e}^{\beta s}(C_s^{\Theta_\varepsilon})^2 \mathrm{d}s\right] \\
&= \mathbb{E}\left[\mathrm{e}^{\beta T}(\bar{V}_T^{\varepsilon})^2\right] - \beta \mathbb{E}\left[\int_t^T \mathrm{e}^{\beta s}(\bar{V}_s^{\varepsilon})^2 \mathrm{d}s\right] - 2\mathbb{E}\left[\int_t^T \mathrm{e}^{\beta s}\bar{V}_s^{\varepsilon}\alpha_s^{\varepsilon} \mathrm{d}s\right] \\
&\leq \mathbb{E}\left[\mathrm{e}^{\beta T}(\bar{V}_T^{\varepsilon})^2\right] + (k-\beta)\mathbb{E}\left[\int_t^T \mathrm{e}^{\beta s}(\bar{V}_s^{\varepsilon})^2 \mathrm{d}s\right] + \frac{1}{k}\mathbb{E}\left[\int_t^T \mathrm{e}^{\beta s}(\alpha_s^{\varepsilon})^2 \mathrm{d}s\right], \quad (37)
\end{aligned}
$$

where we used property (32). The combination of (37) with Lemma 2 provides

$$
\begin{aligned}
&\mathbb{E}\left[\mathrm{e}^{\beta t}(\bar{V}_t^{\varepsilon})^2\right] + (\beta - k)\mathbb{E}\left[\int_t^T \mathrm{e}^{\beta s}(\bar{V}_s^{\varepsilon})^2 \mathrm{d}s\right] + \left(1 - \frac{C}{k}\right)\mathbb{E}\left[\int_t^T \mathrm{e}^{\beta s}(\bar{A}_s^{\varepsilon})^2 \mathrm{d}s\right] \\
&+ \left(1 - \frac{C}{k}\right)\mathbb{E}\left[\int_t^T \mathrm{e}^{\beta s}\int_{\mathbb{R}_0}(\bar{B}_s^{\Theta_\varepsilon}(z))^2 \ell(\mathrm{d}z)\mathrm{d}s\right] + \left(1 - \frac{C}{k}\right)\mathbb{E}\left[\int_t^T \mathrm{e}^{\beta s}(C_s^{\Theta_\varepsilon})^2 \mathrm{d}s\right] \\
&\leq \mathbb{E}\left[\mathrm{e}^{\beta T}(\bar{V}_T^{\varepsilon})^2\right] + \frac{C}{k}\widetilde{G}^4(\varepsilon)\left\{\mathbb{E}\left[\int_t^T \mathrm{e}^{\beta s}(A_s^{\Theta_0})^2 \mathrm{d}s\right]\right. \qquad (38) \\
&\quad \left. + \mathbb{E}\left[\int_t^T \mathrm{e}^{\beta s}\int_{\mathbb{R}_0}(B_s^{\Theta_0}(z))^2 \ell(\mathrm{d}z)\mathrm{d}s\right]\right\}.
\end{aligned}
$$

Let us choose k and β such that $1 - \frac{C}{k} \geq \frac{1}{2}$ and $\beta - k \geq \frac{1}{2}$. This means we choose $k \geq 2C > 0$ and $\beta \geq \frac{1}{2} + k > 0$. Thus for any $s \in [t, T]$ it holds that $1 < \mathrm{e}^{\beta s} \leq \mathrm{e}^{\beta T}$. We derive from (38) that

$$
\begin{aligned}
&\mathbb{E}\left[\int_t^T (\bar{V}_s^{\varepsilon})^2 \mathrm{d}s\right] + \mathbb{E}\left[\int_t^T (\bar{A}_s^{\varepsilon})^2 \mathrm{d}s\right] \\
&\quad + \mathbb{E}\left[\int_t^T \int_{\mathbb{R}_0}(\bar{B}_s^{\Theta_\varepsilon}(z))^2 \ell(\mathrm{d}z)\mathrm{d}s\right] + \mathbb{E}\left[\int_t^T (C_s^{\Theta_\varepsilon})^2 \mathrm{d}s\right] \\
&\leq C\left(\mathbb{E}\left[(\bar{V}_T^{\varepsilon})^2\right] + \widetilde{G}^4(\varepsilon)\left\{\mathbb{E}\left[\int_t^T (A_s^{\Theta_0})^2 \mathrm{d}s\right] + \mathbb{E}\left[\int_t^T \int_{\mathbb{R}_0}(B_s^{\Theta_0}(z))^2 \ell(\mathrm{d}z)\mathrm{d}s\right]\right\}\right).
\end{aligned}
$$

By Lemma 1 we conclude the proof. □

This main result leads to the following theorem concerning the robustness of the discounted portfolio value process of the RM strategy.

Theorem 2 *Assume Assumptions 1. Let $\hat{V}^{\Theta_0}$, $\hat{V}^{\Theta_\varepsilon}$ be part of the solution of* (11), (25) *respectively. Then we have*

$$\mathbb{E}\left[\sup_{0\le t\le T}(\hat{V}_s^{\Theta_0}-\hat{V}_s^{\Theta_\varepsilon})^2\right]\le C\left(\mathbb{E}[(\hat{H}_T-\hat{H}_T^\varepsilon)^2]+\widetilde{G}^4(\varepsilon)\mathbb{E}[\hat{H}_T^2]\right),$$

for a positive constant C.

Proof Integration of the BSDEJ (35) results into

$$\bar{V}_t^\varepsilon=\bar{V}_T^\varepsilon-\int_t^T\alpha_s^\varepsilon\mathrm{d}s-\int_t^T\bar{A}_s^\varepsilon\mathrm{d}W_s-\int_t^T\int_{\mathbb{R}_0}\bar{B}_s^\varepsilon(z)\widetilde{N}(\mathrm{d}s,\mathrm{d}z)+\int_t^T C_s^{\Theta_\varepsilon}\mathrm{d}\widetilde{W}_s.$$

By property (33) we arrive at

$$\begin{aligned}&\mathbb{E}\left[\sup_{0\le t\le T}(\bar{V}_t^\varepsilon)^2\right]\\&\le 5\Bigg(\mathbb{E}\left[(\bar{V}_T^\varepsilon)^2\right]+\mathbb{E}\left[\int_0^T(\alpha_s^\varepsilon)^2\mathrm{d}s\right]+\mathbb{E}\left[\sup_{0\le t\le T}\left(\int_t^T\bar{A}_s^\varepsilon\mathrm{d}W_s\right)^2\right]\\&\qquad+\mathbb{E}\left[\sup_{0\le t\le T}\left(\int_t^T\int_{\mathbb{R}_0}\bar{B}_s^\varepsilon(z)\widetilde{N}(\mathrm{d}s,\mathrm{d}z)\right)^2\right]+\mathbb{E}\left[\sup_{0\le t\le T}\left(\int_t^T C_s^{\Theta_\varepsilon}\mathrm{d}\widetilde{W}_s\right)^2\right]\Bigg).\end{aligned}$$

Burkholder's inequality (see e.g., Theorem 3.28 in [15]) guarantees the existence of a positive constant C such that

$$\begin{aligned}\mathbb{E}\left[\sup_{0\le t\le T}\left(\int_t^T\bar{A}_s^\varepsilon\mathrm{d}W_s\right)^2\right]&\le C\mathbb{E}\left[\int_0^T(\bar{A}_s^\varepsilon)^2\mathrm{d}s\right],\\ \mathbb{E}\left[\sup_{0\le t\le T}\left(\int_t^T\int_{\mathbb{R}_0}\bar{B}_s^\varepsilon(z)\widetilde{N}(\mathrm{d}s,\mathrm{d}z)\right)^2\right]&\le C\mathbb{E}\left[\int_0^T\int_{\mathbb{R}_0}(\bar{B}_s^\varepsilon(z))^2\ell(\mathrm{d}z)\mathrm{d}s\right],\\ \mathbb{E}\left[\sup_{0\le t\le T}\left(\int_t^T C_s^{\Theta_\varepsilon}\mathrm{d}\widetilde{W}_s\right)^2\right]&\le C\mathbb{E}\left[\int_0^T(C_s^{\Theta_\varepsilon})^2\mathrm{d}s\right].\end{aligned}$$

Application of Lemma 2 for $t=0$, $\beta=0$, Lemma 1, and Theorem 1 completes the proof. □

3.3 Robustness of the Risk-Minimising Strategy

Theorem 2 in the previous subsection concerns the robustness result of the value process of the discounted portfolio in the RM strategy. Before we present the stability of the amount of wealth in the RM strategy, we study the relation between $\hat{\pi}^{\Theta_0}$ (resp. $\hat{\pi}^{\Theta_\varepsilon}$) and the solution of the BSDEJ of type (11) (resp. (25)) in the first (resp. second) model. Consider the processes A^{Θ_0} and $B^{\Theta_0}(z)$ defined in (12), then it holds that

$$
\begin{aligned}
&A^{\Theta_0} b + \int_{\mathbb{R}_0} B^{\Theta_0}(z)(\mathrm{e}^z - 1)\rho(z;\Theta_0)\ell(\mathrm{d}z) \\
&= \hat{\pi}^{\Theta_0} b^2 + X^{\Theta_0} b + \int_{\mathbb{R}_0} \left(\hat{\pi}^{\Theta_0}(\mathrm{e}^z - 1)^2 \rho(z;\Theta_0) + Y^{\Theta_0}(z)(\mathrm{e}^z - 1)\rho(z;\Theta_0)\right)\ell(\mathrm{d}z) \\
&= \hat{\pi}^{\Theta_0}\left\{ b^2 + \int_{\mathbb{R}_0} (\mathrm{e}^z - 1)^2 \rho(z;\Theta_0)\ell(\mathrm{d}z)\right\} \\
&\quad + X^{\Theta_0} b + \int_{\mathbb{R}_0} Y^{\Theta_0}(z)(\mathrm{e}^z - 1)\rho(z;\Theta_0)\ell(\mathrm{d}z).
\end{aligned}
$$

From property (10) we attain that

$$
\hat{\pi}^{\Theta_0} = \frac{1}{\kappa_0}\left(A^{\Theta_0} b + \int_{\mathbb{R}_0} B^{\Theta_0}(z)(\mathrm{e}^z - 1)\rho(z;\Theta_0)\ell(\mathrm{d}z)\right), \tag{39}
$$

where $\kappa_0 = b^2 + \int_{\mathbb{R}_0}(\mathrm{e}^z - 1)^2\rho(z;\Theta_0)\ell(\mathrm{d}z)$. Similarly for the second setting we have for the processes A^{Θ_ε}, $B^{\Theta_\varepsilon}(z)$, and C^{Θ_ε} defined in (26) that

$$
\begin{aligned}
&A^{\Theta_\varepsilon} b + \int_{\{|z|\geq\varepsilon\}} B^{\Theta_\varepsilon}(z)(\mathrm{e}^z - 1)\rho(z;\Theta_\varepsilon)\ell(\mathrm{d}z) + C^{\Theta_\varepsilon} G(\varepsilon) \\
&= \hat{\pi}^{\Theta_\varepsilon}\left\{ b^2 + \int_{\{|z|\geq\varepsilon\}} (\mathrm{e}^z - 1)^2\rho(z;\Theta_\varepsilon)\ell(\mathrm{d}z) + G^2(\varepsilon)\right\} \\
&\quad + X^{\Theta_\varepsilon} b + \int_{\{|z|\geq\varepsilon\}} Y^{\Theta_\varepsilon}(z)(\mathrm{e}^z - 1)\rho(z;\Theta_\varepsilon)\ell(\mathrm{d}z) + Z^{\Theta_\varepsilon} G(\varepsilon).
\end{aligned}
$$

Property (24) leads to

$$
\hat{\pi}^{\Theta_\varepsilon} = \frac{1}{\kappa_\varepsilon}\left(A^{\Theta_\varepsilon} b + \int_{\{|z|\geq\varepsilon\}} B^{\Theta_\varepsilon}(z)(\mathrm{e}^z - 1)\rho(z;\Theta_\varepsilon)\ell(\mathrm{d}z) + C^{\Theta_\varepsilon} G(\varepsilon)\right), \tag{40}
$$

where $\kappa_\varepsilon = b^2 + \int_{\{|z|\geq\varepsilon\}}(\mathrm{e}^z - 1)^2\rho(z;\Theta_\varepsilon)\ell(\mathrm{d}z) + G^2(\varepsilon)$.

We introduce the following additional assumption on the Lévy measure which we need for the robustness results studied later.

Assumption 2 For the Lévy measure ℓ the following integrability condition holds

$$
\int_{\{z\geq 1\}} \mathrm{e}^{4z}\ell(\mathrm{d}z) < \infty.
$$

Note that the latter assumption, combined with (30), implies for $k \in \{2, 4\}$ that

$$\int_{\mathbb{R}_0} (e^z - 1)^k \ell(dz) \leq C \left(\int_{\{|z|<1\}} z^2 \ell(dz) + \int_{\{|z|\geq 1\}} \ell(dz) + \int_{\{z \geq 1\}} e^{4z} \ell(dz) \right) < \infty. \tag{41}$$

Moreover Assumption 2 is fulfilled for the considered martingale measures described in Sect. 3.1. Indeed, consider the ET, applying (27) for $\theta = 4$ and restricting the integral over $\{z \geq 1\}$ implies Assumption 2. On the set $\{z \geq 1\}$ it holds that $z \leq e^z - 1$ and therefore Assumption 2 follows from (28) by choosing $\theta = 4$. For the MMM, condition (29) corresponds exactly to Assumption 2.

Theorem 3 *Impose Assumptions 1 and 2. Let the processes $\hat{\pi}^{\Theta_0}$ and $\hat{\pi}^{\Theta_\varepsilon}$ denote the amounts of wealth in a RM strategy. There is a positive constant C such that for any $t \in [0, T]$*

$$\mathbb{E}\left[\int_t^T (\hat{\pi}_s^{\Theta_0} - \hat{\pi}_s^{\Theta_\varepsilon})^2 ds\right] \leq C \left(\mathbb{E}[(\hat{H}_T - \hat{H}_T^\varepsilon)^2] + \widetilde{G}^4(\varepsilon)\mathbb{E}[\hat{H}_T^2] \right).$$

Proof Consider the amounts of wealth in (39) and (40). Let us denote $\hat{\pi}^{\Theta_0} = \frac{1}{\kappa_0}\Upsilon^0$ and $\hat{\pi}^{\Theta_\varepsilon} = \frac{1}{\kappa_\varepsilon}\Upsilon^\varepsilon$. Then it holds that

$$\left(\hat{\pi}^{\Theta_0} - \hat{\pi}^{\Theta_\varepsilon}\right)^2 \leq 2\left(\left(\frac{\kappa_0 - \kappa_\varepsilon}{\kappa_0 \kappa_\varepsilon} \right)^2 (\Upsilon^0)^2 + \frac{1}{\kappa_\varepsilon^2} (\Upsilon^0 - \Upsilon^\varepsilon)^2 \right).$$

Herein we have because of the Hölder's inequality, (14), (41), and properties (iv) and (vii) in Assumptions 1 that

$$\begin{aligned}
\left(\frac{\kappa_0 - \kappa_\varepsilon}{\kappa_0 \kappa_\varepsilon} \right)^2 &\leq \frac{3}{b^8} \left(\left(\int_{\{|z|<\varepsilon\}} (e^z - 1)^2 \rho(z; \Theta_0) \ell(dz) \right)^2 \right. \\
&\quad \left. + \left(\int_{\{|z|\geq\varepsilon\}} (e^z - 1)^2 (\rho(z; \Theta_0) - \rho(z; \Theta_\varepsilon)) \ell(dz) \right)^2 + G^4(\varepsilon) \right) \\
&\leq \frac{3}{b^8} \left(C \left(\int_{\{|z|<\varepsilon\}} (e^z - 1)^2 \ell(dz) \right)^2 \right. \\
&\quad \left. + \int_{\mathbb{R}_0} (e^z - 1)^4 \ell(dz) \int_{\mathbb{R}_0} (\rho(z; \Theta_0) - \rho(z; \Theta_\varepsilon))^2 \ell(dz) + G^4(\varepsilon) \right) \\
&\leq C \widetilde{G}^4(\varepsilon).
\end{aligned}$$

On the other hand it is clear from (39) and (40) that

$$
\begin{aligned}
&(\Upsilon^0 - \Upsilon^\varepsilon)^2 \\
&\le 3\left((\bar{A}^\varepsilon)^2 b^2 + (C^{\Theta_\varepsilon})^2 G^2(\varepsilon)\right. \\
&\qquad \left. + \left(\int_{\mathbb{R}_0} (B^{\Theta_0}(z)(\mathrm{e}^z - 1)\rho(z;\Theta_0) - 1_{\{|z|\ge\varepsilon\}} B^{\Theta_\varepsilon}(z)(\mathrm{e}^z - 1)\rho(z;\Theta_\varepsilon))\ell(\mathrm{d}z)\right)^2\right).
\end{aligned}
$$

Herein we derive via Hölder's inequality, (30), (41), and points (iv), (v), and (vii) in Assumptions 1 that

$$
\begin{aligned}
&\left(\int_{\mathbb{R}_0} (B^{\Theta_0}(z)(\mathrm{e}^z - 1)\rho(z;\Theta_0) - 1_{\{|z|\ge\varepsilon\}} B^{\Theta_\varepsilon}(z)(\mathrm{e}^z - 1)\rho(z;\Theta_\varepsilon))\ell(\mathrm{d}z)\right)^2 \\
&= \left(\int_{\mathbb{R}_0} (B^{\Theta_0}(z)(\rho(z;\Theta_0) - \rho(z;\Theta_\varepsilon))(\mathrm{e}^z - 1) + \bar{B}^\varepsilon(z)\rho(z;\Theta_\varepsilon)(\mathrm{e}^z - 1))\ell(\mathrm{d}z)\right)^2 \\
&\le \int_{\mathbb{R}_0} (B^{\Theta_0}(z))^2 \ell(\mathrm{d}z) \int_{\mathbb{R}_0} (\rho(z;\Theta_0) - \rho(z;\Theta_\varepsilon))^2 (\mathrm{e}^z - 1)^2 \ell(\mathrm{d}z) \\
&\quad + \int_{\mathbb{R}_0} (\bar{B}^\varepsilon(z))^2 \ell(\mathrm{d}z) \int_{\mathbb{R}_0} \rho^2(z;\Theta_\varepsilon)(\mathrm{e}^z - 1)^2 \ell(\mathrm{d}z) \\
&\le \int_{\mathbb{R}_0} (B^{\Theta_0}(z))^2 \ell(\mathrm{d}z) \left(\int_{\mathbb{R}_0} (\rho(z;\Theta_0) - \rho(z;\Theta_\varepsilon))^4 \ell(\mathrm{d}z)\right)^{\frac{1}{2}} \left(\int_{\mathbb{R}_0} (\mathrm{e}^z - 1)^4 \ell(\mathrm{d}z)\right)^{\frac{1}{2}} \\
&\quad + \int_{\mathbb{R}_0} (\bar{B}^\varepsilon(z))^2 \ell(\mathrm{d}z) \left(\int_{\{|z|\ge 1\}} \rho^4(z;\Theta_\varepsilon)\ell(\mathrm{d}z) \int_{\{|z|\ge 1\}} (\mathrm{e}^z - 1)^4 \ell(\mathrm{d}z)\right)^{\frac{1}{2}} \\
&\quad + C \int_{\mathbb{R}_0} (\bar{B}^\varepsilon(z))^2 \ell(\mathrm{d}z) \int_{\{|z|<1\}} z^2 \ell(\mathrm{d}z) \\
&\le C\widetilde{G}^4(\varepsilon) \int_{\mathbb{R}_0} (B^{\Theta_0}(z))^2 \ell(\mathrm{d}z) + C \int_{\mathbb{R}_0} (\bar{B}^\varepsilon(z))^2 \ell(\mathrm{d}z).
\end{aligned}
$$

The results above show that

$$
\begin{aligned}
\left(\hat{\pi}_t^{\Theta_0} - \hat{\pi}_t^{\Theta_\varepsilon}\right)^2 \le C \Bigg(&(\bar{A}_t^\varepsilon)^2 + \int_{\mathbb{R}_0} (\bar{B}_t^\varepsilon(z))^2 \ell(\mathrm{d}z) + (C_t^{\Theta_\varepsilon})^2 \\
&+ \widetilde{G}^4(\varepsilon) \left\{ (A_t^{\Theta_0})^2 + \int_{\mathbb{R}_0} (B_t^{\Theta_0}(z))^2 \ell(\mathrm{d}z) \right\} \Bigg).
\end{aligned}
$$

Therefore

$$\begin{aligned}
&\mathbb{E}\left[\int_t^T (\hat{\pi}_s^{\Theta_0} - \hat{\pi}_s^{\Theta_\varepsilon})^2 \mathrm{d}s\right] \\
&\leq C\Bigg(\mathbb{E}\left[\int_t^T (\bar{A}_s^\varepsilon)^2 \mathrm{d}s\right] + \mathbb{E}\left[\int_t^T \int_{\mathbb{R}_0} (\bar{B}_s^\varepsilon(z))^2 \ell(\mathrm{d}z)\mathrm{d}s\right] + \mathbb{E}\left[\int_t^T (C_s^{\Theta_\varepsilon})^2 \mathrm{d}s\right] \\
&\qquad + \widetilde{G}^4(\varepsilon)\left\{\mathbb{E}\left[\int_t^T (A_s^{\Theta_0})^2 \mathrm{d}s\right] + \mathbb{E}\left[\int_t^T \int_{\mathbb{R}_0} (B_s^{\Theta_0}(z))^2 \ell(\mathrm{d}z)\mathrm{d}s\right]\right\}\Bigg).
\end{aligned}$$

By Lemma 1 and Theorem 1 we conclude the proof. □

The trading in the risky assets is gathered in the gain processes defined by $\hat{G}_t^{\Theta_0} = \int_0^t \xi_s^{\Theta_0} \mathrm{d}\hat{S}_s$ and $\hat{G}_t^{\Theta_\varepsilon} = \int_0^t \xi_s^{\Theta_\varepsilon} \mathrm{d}\hat{S}_s^\varepsilon$. The following theorem shows the robustness of this gain process.

Theorem 4 *Under Assumptions 1 and 2, there exists a positive constant C such that for any $t \in [0, T]$*

$$\mathbb{E}\left[\left(\hat{G}_t^{\Theta_0} - \hat{G}_t^{\Theta_\varepsilon}\right)^2\right] \leq C\left(\mathbb{E}[(\hat{H}_T - \hat{H}_T^\varepsilon)^2] + \widetilde{G}^2(\varepsilon)\mathbb{E}[\hat{H}_T^2]\right).$$

Proof From (5) and (6) we know that

$$\begin{aligned}
\xi_s^{\Theta_0}\mathrm{d}\hat{S}_s &= \xi_s^{\Theta_0}\hat{S}_s b\mathrm{d}W_s^{\Theta_0} + \xi_s^{\Theta_0}\hat{S}_s \int_{\mathbb{R}_0} (\mathrm{e}^z - 1)\widetilde{N}^{\Theta_0}(\mathrm{d}s, \mathrm{d}z) \\
&= \hat{\pi}_s^{\Theta_0}\Bigg(\left(-b^2\Theta_0 + \int_{\mathbb{R}_0} (\mathrm{e}^z - 1)\,(1 - \rho(z; \Theta_0))\,\ell(\mathrm{d}z)\right)\mathrm{d}s \\
&\qquad + b\mathrm{d}W_s + \int_{\mathbb{R}_0} (\mathrm{e}^z - 1)\widetilde{N}(\mathrm{d}s, \mathrm{d}z)\Bigg).
\end{aligned}$$

In the other setting we have from (19) and (20) that

$$\begin{aligned}
\xi_s^{\Theta_\varepsilon}\mathrm{d}\hat{S}_s^\varepsilon &= \xi_s^{\Theta_\varepsilon}\hat{S}_s^\varepsilon b\mathrm{d}W_s^{\Theta_\varepsilon} + \xi_s^{\Theta_\varepsilon}\hat{S}_s^\varepsilon \int_{\{|z|\geq\varepsilon\}} (\mathrm{e}^z - 1)\widetilde{N}^{\Theta_\varepsilon}(\mathrm{d}s, \mathrm{d}z) + \xi_s^{\Theta_\varepsilon}\hat{S}_s^\varepsilon G(\varepsilon)\mathrm{d}\widetilde{W}_s^{\Theta_\varepsilon} \\
&= \hat{\pi}_s^{\Theta_\varepsilon}\Bigg(\left(-b^2\Theta_\varepsilon + \int_{\{|z|\geq\varepsilon\}} (\mathrm{e}^z - 1)\,(1 - \rho(z; \Theta_\varepsilon))\,\ell(\mathrm{d}z) - G^2(\varepsilon)\Theta_\varepsilon\right)\mathrm{d}s \\
&\qquad + b\mathrm{d}W_s + \int_{\{|z|\geq\varepsilon\}} (\mathrm{e}^z - 1)\widetilde{N}(\mathrm{d}s, \mathrm{d}z) + G(\varepsilon)\mathrm{d}\widetilde{W}_s\Bigg).
\end{aligned}$$

We derive from the previous SDEs that

$$
\begin{aligned}
&\hat{G}_t^{\Theta_0} - \hat{G}_t^{\Theta_\varepsilon} = \int_0^t \xi_s^{\Theta_0} \mathrm{d}\hat{S}_s - \int_0^t \xi_s^{\Theta_\varepsilon} \mathrm{d}\hat{S}_s^\varepsilon \\
&= \left(-b^2\Theta_0 + \int_{\mathbb{R}_0} (\mathrm{e}^z - 1)\,(1 - \rho(z;\Theta_0))\,\ell(\mathrm{d}z)\right) \int_0^t \hat{\pi}_s^{\Theta_0} \mathrm{d}s \\
&\quad - \left(-b^2\Theta_\varepsilon + \int_{\{|z|\geq\varepsilon\}} (\mathrm{e}^z - 1)\,(1 - \rho(z;\Theta_\varepsilon))\,\ell(\mathrm{d}z) - G^2(\varepsilon)\Theta_\varepsilon\right) \int_0^t \hat{\pi}_s^{\Theta_\varepsilon} \mathrm{d}s \\
&\quad + b\int_0^t (\hat{\pi}_s^{\Theta_0} - \hat{\pi}_s^{\Theta_\varepsilon}) \mathrm{d}W_s + \int_0^t \int_{\mathbb{R}_0} \left(\hat{\pi}_s^{\Theta_0}(\mathrm{e}^z - 1) - \hat{\pi}_s^{\Theta_\varepsilon} 1_{\{|z|\geq\varepsilon\}}(\mathrm{e}^z - 1)\right) \widetilde{N}(\mathrm{d}s, \mathrm{d}z) \\
&\quad - G(\varepsilon)\int_0^t \hat{\pi}_s^{\Theta_\varepsilon} \mathrm{d}\widetilde{W}_s.
\end{aligned}
$$

Via the Cauchy-Schwartz inequality and the Itô isometry we obtain that

$$
\begin{aligned}
&\mathbb{E}\left[\left(\hat{G}_t^{\Theta_0} - \hat{G}_t^{\Theta_\varepsilon}\right)^2\right] \\
&\leq C\Bigg(\mathbb{E}\left[\int_0^t (\hat{\pi}_s^{\Theta_0})^2 \mathrm{d}s\right] \Bigg\{\left(-b^2\Theta_0 + \int_{\mathbb{R}_0} (\mathrm{e}^z - 1)\,(1 - \rho(z;\Theta_0))\,\ell(\mathrm{d}z)\right) \\
&\quad - \left(-b^2\Theta_\varepsilon + \int_{\{|z|\geq\varepsilon\}} (\mathrm{e}^z - 1)\,(1 - \rho(z;\Theta_\varepsilon))\,\ell(\mathrm{d}z) - G^2(\varepsilon)\Theta_\varepsilon\right)\Bigg\}^2 \\
&\quad + \mathbb{E}\left[\int_0^t (\hat{\pi}_s^{\Theta_0} - \hat{\pi}_s^{\Theta_\varepsilon})^2 \mathrm{d}s\right] \\
&\quad \times \left(-b^2\Theta_\varepsilon + \int_{\{|z|\geq\varepsilon\}} (\mathrm{e}^z - 1)\,(1 - \rho(z;\Theta_\varepsilon))\,\ell(\mathrm{d}z) - G^2(\varepsilon)\Theta_\varepsilon\right)^2 \\
&\quad + b^2\mathbb{E}\left[\int_0^t (\hat{\pi}_s^{\Theta_0} - \hat{\pi}_s^{\Theta_\varepsilon})^2 \mathrm{d}s\right] + G^2(\varepsilon)\mathbb{E}\left[\int_0^t (\hat{\pi}_s^{\Theta_\varepsilon})^2 \mathrm{d}s\right] \\
&\quad + \mathbb{E}\left[\int_0^t \int_{\mathbb{R}_0} \left(\hat{\pi}_s^{\Theta_0}(\mathrm{e}^z - 1) - \hat{\pi}_s^{\Theta_\varepsilon} 1_{\{|z|\geq\varepsilon\}}(\mathrm{e}^z - 1)\right)^2 \ell(\mathrm{d}z)\mathrm{d}s\right]\Bigg),
\end{aligned}
$$

wherein

$$
\begin{aligned}
&\mathbb{E}\left[\int_0^t \int_{\mathbb{R}_0} \left(\hat{\pi}_s^{\Theta_0}(\mathrm{e}^z - 1) - \hat{\pi}_s^{\Theta_\varepsilon} 1_{\{|z|\geq\varepsilon\}}(\mathrm{e}^z - 1)\right)^2 \ell(\mathrm{d}z)\mathrm{d}s\right] \\
&\leq 2\mathbb{E}\left[\int_0^t \int_{\mathbb{R}_0} \left((\hat{\pi}_s^{\Theta_0})^2(\mathrm{e}^z - 1)^2 1_{\{|z|<\varepsilon\}} + (\hat{\pi}_s^{\Theta_0} - \hat{\pi}_s^{\Theta_\varepsilon})^2(\mathrm{e}^z - 1)^2 1_{\{|z|\geq\varepsilon\}}\right) \ell(\mathrm{d}z)\mathrm{d}s\right] \\
&\leq 2\Bigg(\int_{\{|z|<\varepsilon\}} (\mathrm{e}^z - 1)^2 \ell(\mathrm{d}z)\mathbb{E}\left[\int_0^t (\hat{\pi}_s^{\Theta_0})^2 \mathrm{d}s\right] \\
&\qquad + \int_{\mathbb{R}_0} (\mathrm{e}^z - 1)^2 \ell(\mathrm{d}z)\mathbb{E}\left[\int_0^t (\hat{\pi}_s^{\Theta_0} - \hat{\pi}_s^{\Theta_\varepsilon})^2 \mathrm{d}s\right]\Bigg),
\end{aligned}
$$

and

$$\mathbb{E}\left[\int_0^t (\hat{\pi}_s^{\Theta_\varepsilon})^2 \mathrm{d}s\right] \le 2\mathbb{E}\left[\int_0^t (\hat{\pi}_s^{\Theta_0} - \hat{\pi}_s^{\Theta_\varepsilon})^2 \mathrm{d}s\right] + 2\mathbb{E}\left[\int_0^t (\hat{\pi}_s^{\Theta_0})^2 \mathrm{d}s\right].$$

By relation (14), Assumptions 1, (39), (41), Lemma 1, and Theorem 3 we prove the statement. □

The following result shows the robustness of the process $\mathscr{L}^{\Theta}$ appearing in the GKW-decomposition. This plays an important role in the stability of the cost process of the RM strategy.

Theorem 5 *Let Assumptions 1 and 2 hold true. Let the processes $\mathscr{L}^{\Theta_0}$ and $\mathscr{L}^{\Theta_\varepsilon}$ be as in (9) and (23), respectively. For any $t \in [0, T]$ it holds that*

$$\mathbb{E}[(\mathscr{L}_t^{\Theta_0} - \mathscr{L}_t^{\Theta_\varepsilon})^2] \le C\left(\mathbb{E}[(\hat{H}_T - \hat{H}_T^\varepsilon)^2] + \widetilde{G}^2(\varepsilon)\mathbb{E}[\hat{H}_T^2]\right),$$

for a positive constant C.

Proof By (5) we can rewrite (9) as

$$\begin{aligned}\mathrm{d}\mathscr{L}_t^{\Theta_0} &= \left(-b\Theta_0 X_t^{\Theta_0} + \int_{\mathbb{R}_0} Y_t^{\Theta_0}(z)(1-\rho(z;\Theta_0))\ell(\mathrm{d}z)\right)\mathrm{d}t \\ &\quad + X_t^{\Theta_0}\mathrm{d}W_t + \int_{\mathbb{R}_0} Y_t^{\Theta_0}(z)\widetilde{N}(\mathrm{d}t,\mathrm{d}z).\end{aligned}$$

and similarly by (19) we obtain for (23)

$$\begin{aligned}\mathrm{d}\mathscr{L}_t^{\Theta_\varepsilon} &= \left(-b\Theta_\varepsilon X_t^{\Theta_\varepsilon} + \int_{\{|z|\ge\varepsilon\}} Y_t^{\Theta_\varepsilon}(z)(1-\rho(z;\Theta_\varepsilon))\ell(\mathrm{d}z) - G(\varepsilon)\Theta_\varepsilon Z_t^{\Theta_\varepsilon}\right)\mathrm{d}t \\ &\quad + X_t^{\Theta_\varepsilon}\mathrm{d}W_t + \int_{\{|z|\ge\varepsilon\}} Y_t^{\Theta_\varepsilon}(z)\widetilde{N}(\mathrm{d}t,\mathrm{d}z) + Z_t^{\Theta_\varepsilon}\mathrm{d}\widetilde{W}_t.\end{aligned}$$

Hence we recover that

$$\mathrm{d}(\mathscr{L}_t^{\Theta_0} - \mathscr{L}_t^{\Theta_\varepsilon}) = \gamma_t^\varepsilon \mathrm{d}t + \bar{X}_t^\varepsilon \mathrm{d}W_t + \int_{\mathbb{R}_0} \bar{Y}_t^\varepsilon(z)\widetilde{N}(\mathrm{d}t,\mathrm{d}z) - Z_t^{\Theta_\varepsilon}\mathrm{d}\widetilde{W}_t,$$

where

$$\begin{aligned}\gamma^\varepsilon &= -b(\Theta_0 X^{\Theta_0} - \Theta_\varepsilon X^{\Theta_\varepsilon}) + G(\varepsilon)\Theta_\varepsilon Z^{\Theta_\varepsilon} \\ &\quad + \int_{\mathbb{R}_0} \left(Y^{\Theta_0}(z)(1-\rho(z;\Theta_0)) - 1_{\{|z|\ge\varepsilon\}} Y^{\Theta_\varepsilon}(z)(1-\rho(z;\Theta_\varepsilon))\right)\ell(\mathrm{d}z), \\ \bar{X}^\varepsilon &= X^{\Theta_0} - X^{\Theta_\varepsilon}, \\ \bar{Y}^\varepsilon(z) &= Y^{\Theta_0}(z) - 1_{\{|z|\ge\varepsilon\}} Y^{\Theta_\varepsilon}(z).\end{aligned}$$

By integration over $[0, t]$ and taking the square we retrieve using (33) that

$$(\mathscr{L}_t^{\Theta_0} - \mathscr{L}_t^{\Theta_\varepsilon})^2 \leq C\Bigg(\left(\int_0^t \gamma_s^\varepsilon \mathrm{d}s\right)^2 + \left(\int_0^t \bar{X}_s^\varepsilon \mathrm{d}W_s\right)^2 + \left(\int_0^t \int_{\mathbb{R}_0} \bar{Y}_s^\varepsilon(z)\widetilde{N}(\mathrm{d}s, \mathrm{d}z)\right)^2 + \left(\int_0^t Z_s^{\Theta_\varepsilon}\mathrm{d}\widetilde{W}_s\right)^2\Bigg).$$

Via the Cauchy-Schwartz inequality and the Itô isometry it follows that

$$\mathbb{E}[(\mathscr{L}_t^{\Theta_0} - \mathscr{L}_t^{\Theta_\varepsilon})^2] \leq C\Bigg(\mathbb{E}\left[\int_0^t (\gamma_s^\varepsilon)^2 \mathrm{d}s\right] + \mathbb{E}\left[\int_0^t (\bar{X}_s^\varepsilon)^2 \mathrm{d}s\right] + \mathbb{E}\left[\int_0^t \int_{\mathbb{R}_0} (\bar{Y}_s^\varepsilon(z))^2 \ell(\mathrm{d}z)\mathrm{d}s\right] + \mathbb{E}\left[\int_0^t (Z_s^{\Theta_\varepsilon})^2 \mathrm{d}s\right]\Bigg).$$

Concerning the term $\mathbb{E}\left[\int_0^t (\gamma_s^\varepsilon)^2 \mathrm{d}s\right]$ we derive through (ii) and (iii) in Assumptions 1 that

$$\begin{aligned}
&\mathbb{E}\left[\int_0^t \left(\Theta_0 X_s^{\Theta_0} - \Theta_\varepsilon X_s^{\Theta_\varepsilon}\right)^2 \mathrm{d}s\right] \\
&\leq 2\left(\mathbb{E}\left[\int_0^t (\Theta_0 - \Theta_\varepsilon)^2 (X_s^{\Theta_0})^2 \mathrm{d}s\right] + \mathbb{E}\left[\int_0^t {\Theta_\varepsilon}^2 (X_s^{\Theta_0} - X_s^{\Theta_\varepsilon})^2 \mathrm{d}s\right]\right) \\
&\leq C\left(\widetilde{G}^4(\varepsilon)\mathbb{E}\left[\int_0^t (X_s^{\Theta_0})^2 \mathrm{d}s\right] + \mathbb{E}\left[\int_0^t (\bar{X}_s^\varepsilon)^2 \mathrm{d}s\right]\right)
\end{aligned}$$

and via (vi) and (vii) in Assumptions 1 it follows that

$$\begin{aligned}
&\mathbb{E}\left[\int_0^t \left\{\int_{\mathbb{R}_0} \left(Y_s^{\Theta_0}(z)\,(1 - \rho(z;\Theta_0)) - 1_{\{|z|\geq\varepsilon\}} Y_s^{\Theta_\varepsilon}(z)\,(1 - \rho(z;\Theta_\varepsilon))\right)\ell(\mathrm{d}z)\right\}^2 \mathrm{d}s\right] \\
&\leq \int_{\mathbb{R}_0} (\rho(z;\Theta_0) - \rho(z;\Theta_\varepsilon))^2\, \ell(\mathrm{d}z)\mathbb{E}\left[\int_0^t \int_{\mathbb{R}_0} (Y_s^{\Theta_0}(z))^2 \ell(\mathrm{d}z)\mathrm{d}s\right] \\
&\quad + \int_{\mathbb{R}_0} (1 - \rho(z;\Theta_\varepsilon))^2\, \ell(\mathrm{d}z)\mathbb{E}\left[\int_0^t \int_{\mathbb{R}_0} (\bar{Y}_s^\varepsilon(z))^2 \ell(\mathrm{d}z)\mathrm{d}s\right] \\
&\leq C\left(\widetilde{G}^4(\varepsilon)\mathbb{E}\left[\int_0^t \int_{\mathbb{R}_0} (Y_s^{\Theta_0}(z))^2 \ell(\mathrm{d}z)\mathrm{d}s\right] + \mathbb{E}\left[\int_0^t \int_{\mathbb{R}_0} (\bar{Y}_s^\varepsilon(z))^2 \ell(\mathrm{d}z)\mathrm{d}s\right]\right).
\end{aligned}$$

Thus we obtain that

$$
\begin{aligned}
&\mathbb{E}[(\mathscr{L}_t^{\Theta_0} - \mathscr{L}_t^{\Theta_\varepsilon})^2] \\
&\leq C\Bigg(\widetilde{G}^4(\varepsilon)\left\{\mathbb{E}\left[\int_0^t (X_s^{\Theta_0})^2 \mathrm{d}s\right] + \mathbb{E}\left[\int_0^t \int_{\mathbb{R}_0} (Y_s^{\Theta_0}(z))^2 \ell(\mathrm{d}z)\mathrm{d}s\right]\right\} \\
&\quad + \mathbb{E}\left[\int_0^t (\bar{X}_s^{\varepsilon})^2 \mathrm{d}s\right] + \mathbb{E}\left[\int_0^t \int_{\mathbb{R}_0} (\bar{Y}_s^{\varepsilon}(z))^2 \ell(\mathrm{d}z)\mathrm{d}s\right] + \mathbb{E}\left[\int_0^t (Z_s^{\Theta_\varepsilon})^2 \mathrm{d}s\right]\Bigg).
\end{aligned}
\tag{42}
$$

Let us consider the terms appearing in the latter expression separately.

- Definition (12) implies that

$$
\mathbb{E}\left[\int_0^t (X_s^{\Theta_0})^2 \mathrm{d}s\right] \leq 2\left(\mathbb{E}\left[\int_0^t (A_s^{\Theta_0})^2 \mathrm{d}s\right] + b^2 \mathbb{E}\left[\int_0^t (\hat{\pi}_s^{\Theta_0})^2 \mathrm{d}s\right]\right)
$$

  and

$$
\begin{aligned}
&\mathbb{E}\left[\int_0^t \int_{\mathbb{R}_0} (Y_s^{\Theta_0}(z))^2 \ell(\mathrm{d}z)\mathrm{d}s\right] \\
&\leq 2\left(\mathbb{E}\left[\int_0^t \int_{\mathbb{R}_0} (B_s^{\Theta_0}(z))^2 \ell(\mathrm{d}z)\mathrm{d}s\right] + \int_{\mathbb{R}_0} (\mathrm{e}^z - 1)^2 \ell(\mathrm{d}z)\mathbb{E}\left[\int_0^t (\hat{\pi}_s^{\Theta_0})^2 \mathrm{d}s\right]\right).
\end{aligned}
$$

- Combining (12) and (26) in

$$
\bar{X}_t^{\varepsilon} = X_t^{\Theta_0} - X_t^{\Theta_\varepsilon} = \bar{A}_t^{\varepsilon} - (\hat{\pi}_t^{\Theta_0} - \hat{\pi}_t^{\Theta_\varepsilon})b,
$$

  it easily follows that

$$
\mathbb{E}\left[\int_0^t (\bar{X}_s^{\varepsilon})^2 \mathrm{d}s\right] \leq C\left(\mathbb{E}\left[\int_0^t (\bar{A}_s^{\varepsilon})^2 \mathrm{d}s\right] + \mathbb{E}\left[\int_0^t (\hat{\pi}_s^{\Theta_0} - \hat{\pi}_s^{\Theta_\varepsilon})^2 \mathrm{d}s\right]\right).
$$

- Similarly, from (12) and (26) we find

$$
\bar{Y}_t^{\varepsilon}(z) = Y_t^{\Theta_0}(z) - Y_t^{\Theta_\varepsilon}(z) = \bar{B}_t^{\varepsilon}(z) - (\hat{\pi}_t^{\Theta_0} - \hat{\pi}_t^{\Theta_\varepsilon})(\mathrm{e}^z - 1).
$$

  Hence

$$
\begin{aligned}
&\mathbb{E}\left[\int_0^t \int_{\mathbb{R}_0} (\bar{Y}_s^{\varepsilon}(z))^2 \ell(\mathrm{d}z)\mathrm{d}s\right] \\
&\leq 2\left(\mathbb{E}\left[\int_0^t \int_{\mathbb{R}_0} (\bar{B}_s^{\varepsilon}(z))^2 \ell(\mathrm{d}z)\mathrm{d}s\right] + \int_{\mathbb{R}_0} (\mathrm{e}^z - 1)^2 \ell(\mathrm{d}z)\mathbb{E}\left[\int_0^t (\hat{\pi}_s^{\Theta_0} - \hat{\pi}_s^{\Theta_\varepsilon})^2 \mathrm{d}s\right]\right).
\end{aligned}
$$

- From (26), the estimate

$$(Z_t^{\Theta_\varepsilon}(z))^2 \le C\left((C_t^{\Theta_\varepsilon})^2 + (\hat{\pi}_t^{\Theta_0} - \hat{\pi}_t^{\Theta_\varepsilon})^2 G^2(\varepsilon) + (\hat{\pi}_t^{\Theta_0})^2 G^2(\varepsilon)\right)$$

leads to

$$\mathbb{E}\left[\int_0^t (Z_s^{\Theta_\varepsilon}(z))^2 \mathrm{d}s\right] \le C\left(\mathbb{E}\left[\int_0^t (C_s^{\Theta_\varepsilon})^2 \mathrm{d}s\right] + G^2(\varepsilon)\mathbb{E}\left[\int_0^t (\hat{\pi}_s^{\Theta_0} - \hat{\pi}_s^{\Theta_\varepsilon})^2 \mathrm{d}s\right]\right.$$
$$\left. + G^2(\varepsilon)\mathbb{E}\left[\int_0^t (\hat{\pi}_s^{\Theta_0})^2 \mathrm{d}s\right]\right).$$

- Because of (39) and (vi) in Assumptions 1 we notice that

$$\mathbb{E}\left[\int_0^t (\hat{\pi}_s^{\Theta_0})^2 \mathrm{d}s\right] \le C\left(\mathbb{E}\left[\int_0^t (A_s^{\Theta_0})^2 \mathrm{d}s\right] + \mathbb{E}\left[\int_0^t \int_{\mathbb{R}_0} (B_s^{\Theta_0}(z))^2 \ell(\mathrm{d}z)\mathrm{d}s\right]\right).$$

Using (41) and the combination of the above inequalities in (42) show that

$$\mathbb{E}[(\mathscr{L}_t^{\Theta_0} - \mathscr{L}_t^{\Theta_\varepsilon})^2] \le C\left(\widetilde{G}^2(\varepsilon)\left\{\mathbb{E}\left[\int_0^t (A_s^{\Theta_0})^2 \mathrm{d}s\right] + \mathbb{E}\left[\int_0^t\int_{\mathbb{R}_0} (B_s^{\Theta_0}(z))^2 \ell(\mathrm{d}z)\mathrm{d}s\right]\right\}\right.$$
$$+ \mathbb{E}\left[\int_0^t (\bar{A}_s^\varepsilon)^2 \mathrm{d}s\right] + \mathbb{E}\left[\int_0^t \int_{\mathbb{R}_0} (\bar{B}_s^\varepsilon(z))^2 \ell(\mathrm{d}z)\mathrm{d}s\right]$$
$$\left. + \mathbb{E}\left[\int_0^t (C_s^{\Theta_\varepsilon})^2 \mathrm{d}s\right] + \mathbb{E}\left[\int_0^t (\hat{\pi}_s^{\Theta_0} - \hat{\pi}_s^{\Theta_\varepsilon})^2 \mathrm{d}s\right]\right).$$

Finally by Lemma 1 and Theorems 1 and 3 we conclude the proof. □

The cost processes of the quadratic hedging strategy for $\hat{H}_T$, $\hat{H}_T^\varepsilon$ are defined by $K^{\Theta_0} = \mathscr{L}^{\Theta_0} + \hat{V}_0^{\Theta_0}$ and $K^{\Theta_\varepsilon} = \mathscr{L}^{\Theta_\varepsilon} + \hat{V}_0^{\Theta_\varepsilon}$. The upcoming result concerns the robustness of the cost process and follows directly from the previous theorem.

Corollary 1 *Under Assumptions 1 and 2, there exists a positive constant C such that it holds for all $t \in [0, T]$ that*

$$\mathbb{E}[(K_t^{\Theta_0} - K_t^{\Theta_\varepsilon})^2] \le C\left(\mathbb{E}[(\hat{H}_T - \hat{H}_T^\varepsilon)^2] + \widetilde{G}^2(\varepsilon)\mathbb{E}[\hat{H}_T^2]\right).$$

Proof Notice that

$$\mathbb{E}[(K_t^{\Theta_0} - K_t^{\Theta_\varepsilon})^2] \le 2\left(\mathbb{E}[(\mathscr{L}_t^{\Theta_0} - \mathscr{L}_t^{\Theta_\varepsilon})^2] + \mathbb{E}[(\hat{V}_0^{\Theta_0} - \hat{V}_0^{\Theta_\varepsilon})^2]\right),$$

wherein

$$\mathbb{E}[(\hat{V}_0^{\Theta_0} - \hat{V}_0^{\Theta_\varepsilon})^2] \leq \mathbb{E}\left[\sup_{0\leq t\leq T} (\hat{V}_t^{\Theta_0} - \hat{V}_t^{\Theta_\varepsilon})^2\right].$$

Theorems 2 and 5 complete the proof. □

3.4 Robustness Results for the Mean-Variance Hedging

Since the optimal numbers ξ^{Θ_0} and ξ^{Θ_ε} of risky assets are the same in the RM and the MVH strategy, the amounts of wealth $\hat{\pi}^{\Theta_0}$ and $\hat{\pi}^{\Theta_\varepsilon}$ and the gain processes $\hat{G}^{\Theta_0}$ and $\hat{G}^{\Theta_\varepsilon}$ also coincide for both strategies. Therefore we conclude that the robustness results of the amount of wealth and gain process also hold true for the MVH strategy, see Theorems 3 and 4.

The cost for a MVH strategy is not the same as for the RM strategy. However, under the assumption that a fixed starting amount $\widetilde{V}_0$ is available to set up a MVH strategy, we derive a robustness result for the loss at time of maturity. For the models (1) and (13), it holds that the losses at time of maturity T are given by

$$\begin{aligned} L^{\Theta_0} &= \hat{H}_T - \widetilde{V}_0 - \int_0^T \xi_s^{\Theta_0} \mathrm{d}\hat{S}_s, \\ L^{\Theta_\varepsilon} &= \hat{H}_T^\varepsilon - \widetilde{V}_0 - \int_0^T \xi_s^{\Theta_\varepsilon} \mathrm{d}\hat{S}_s^\varepsilon. \end{aligned}$$

When Assumptions 1 and 2 are imposed, we derive via Theorem 4 that

$$\mathbb{E}[(L^{\Theta_0} - L^{\Theta_\varepsilon})^2] \leq C\left(\mathbb{E}[(\hat{H}_T - \hat{H}_T^\varepsilon)^2] + \widetilde{G}^2(\varepsilon)\mathbb{E}[\hat{H}_T^2]\right),$$

for a positive constant C.

Note that we cannot draw any conclusions from the results above about the robustness of the value of the discounted portfolio for the MVH strategy, since the portfolios are strictly different for both strategies.

4 Conclusion

Two different geometric Lévy stock price models were considered in this paper. We proved that the RM and the MVH strategies in a martingale setting are stable against the choice of the model. To this end the two models were considered under different risk-neutral measures that are dependent on the specific price models. The robustness results are derived through the use of BSDEJs and the obtained L^2-convergence rates are expressed in terms of estimates of the form $\mathbb{E}[(\hat{H}_T - \hat{H}_T^\varepsilon)^2]$. The latter estimate is a well studied quantity, see [3, 16]. In the current paper, we considered two possible models for the price process. Starting from the initial model (1) other models could be constructed by truncating the small jumps and possibly rescaling the original Brownian motion (cfr. [8]). Similar robustness results hold for quadratic hedging strategies in a martingale setting in these other models.

In [8] a semimartingale setting was considered and conditions had to be imposed to guarantee the existence of the solutions to the BSDEJs. In this paper however, we considered a martingale setting and, since there is no driver in the BSDEJs, the existence of the solution to the BSDEJs was immediately guaranteed. On the other hand, since the two models were considered under two different martingale measures, we had to fall back on the common historical measure for the robustness study. Therefore, a robustness study of the martingale measures had to be performed and additional terms made some computations more involved compared to the semi-martingale setting studied in [8].

In this approach based on BSDEJs we could not find explicit robustness results for the optimal number of risky assets. Therefore we refer to [6], where a robustness study is performed in a martingale and semimartingale setting based on Fourier transforms. Note that in [6] robustness was mainly studied in the L^1-sense and the authors noted that their results can be extended into L^2-convergence, whereas L^2-robustness results are explicitly derived in the current paper.

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